Section 12.3 The Dot Product

Problem (12.3.8) For $\mathbf{v} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \rangle$, $\mathbf{u} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \rangle$ find

a) $\mathbf{v} \cdot \mathbf{u}$, $|\mathbf{v}|$, and $|\mathbf{u}|$,
b) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$,
c) the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$,
d) the vector $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{u} &= \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \rangle \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \rangle = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \\
|\mathbf{v}| &= \sqrt{\frac{1}{2} + \frac{1}{3}} = \sqrt{\frac{5}{6}} \\
|\mathbf{u}| &= \sqrt{\frac{1}{2} + \frac{1}{3}} = \sqrt{\frac{5}{6}} \\
\cos\theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{1}{6}\sqrt{\frac{6}{5}}\sqrt{\frac{6}{5}} = \frac{1}{5} \\
\text{scalar component of } \mathbf{u} \text{ in the direction of } \mathbf{v} &= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{1}{6}\sqrt{\frac{6}{5}} = \frac{1}{\sqrt{30}} \\
\text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v} = \left(\frac{1}{6} \times \frac{6}{5}\right)\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \rangle = \langle \frac{1}{5\sqrt{2}}, \frac{1}{5\sqrt{3}} \rangle
\end{aligned}$$

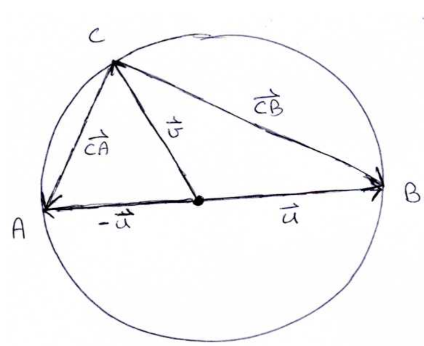
Problem (12.3.18) Write $\mathbf{u} = \mathbf{j} + \mathbf{k}$ as a sum of vector parallel and a vector orthogonal to $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

$$\begin{aligned}
\mathbf{u} &= \mathbf{j} + \mathbf{k} = \langle 0, 1, 1 \rangle, & |\mathbf{u}| &= \sqrt{2} \\
\mathbf{v} &= \mathbf{i} + \mathbf{j} = \langle 1, 1, 0 \rangle, & |\mathbf{v}| &= \sqrt{2}
\end{aligned}$$

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= 1 \\
\text{parallel} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \left(\frac{1}{2}\right)\langle 1, 1, 0 \rangle = \langle \frac{1}{2}, \frac{1}{2}, 0 \rangle \\
\text{orthogonal} &= \mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \langle 0, 1, 1 \rangle - \left(\frac{1}{2}\right)\langle 1, 1, 0 \rangle = \langle -\frac{1}{2}, \frac{1}{2}, 1 \rangle
\end{aligned}$$

There are some nice plots of what this means in the associated *Mathematica* file.

Problem (12.3.22) Suppose AB is the diameter of a circle with center O and that C is a point on one of the two arcs joining A and B. Show $\vec{CA}$ and $\vec{CB}$ are orthogonal.



If we can show that $\vec{CA} \cdot \vec{CB} = 0$, then the two vectors are orthogonal since their dot product is zero.

From the diagram, we have

$$\begin{aligned}
\mathbf{v} + \vec{CA} &= -\mathbf{u} \Rightarrow \vec{CA} = -\mathbf{v} - \mathbf{u} \\
\mathbf{v} + \vec{CB} &= -\mathbf{v} \Rightarrow \vec{CB} = -\mathbf{v} + \mathbf{u} \\
\vec{CA} \cdot \vec{CB} &= (-\mathbf{v} - \mathbf{u}) \cdot (-\mathbf{v} + \mathbf{u}) \\
&= \mathbf{v} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{u} \\
&= |\mathbf{v}|^2 + \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - |\mathbf{u}|^2 \\
&= |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0 \text{ since the length of } \mathbf{v} \text{ is the same as the length of } \mathbf{u}
\end{aligned}$$

Since the dot product is zero, the vectors are orthogonal.

Problem (12.3.32) If $\mathbf{u} \cdot \mathbf{v}_1 = \mathbf{u} \cdot \mathbf{v}_2$ and $\mathbf{u} \neq \mathbf{0}$, can you conclude that $\mathbf{v}_1 = \mathbf{v}_2$? Give reasons for your answer.

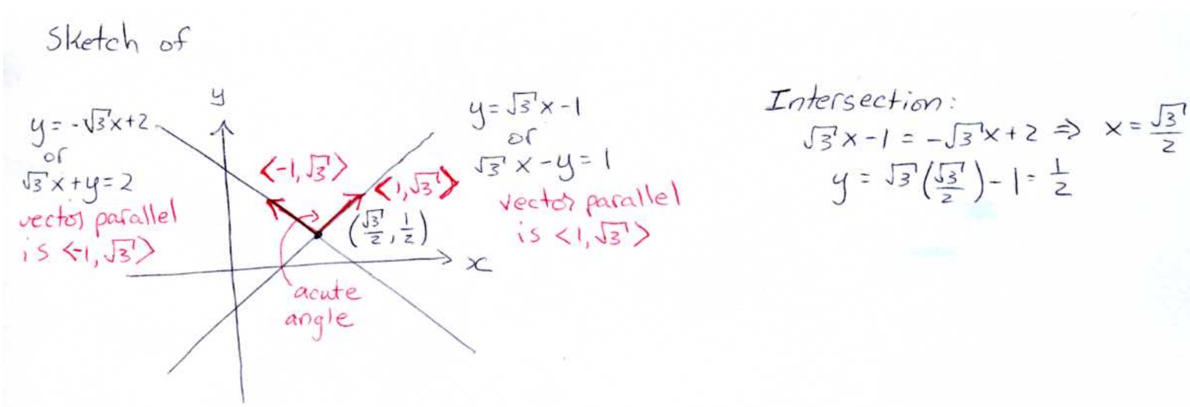
The MMA file contains a numerical counter example. Here is another solution.

If we are given $\mathbf{u} \cdot \mathbf{v}_1 = \mathbf{u} \cdot \mathbf{v}_2$, then we have $\mathbf{u} \cdot (\mathbf{v}_1 - \mathbf{v}_2) = 0$. This tells us that $\mathbf{u}$ is perpendicular to $\mathbf{v}_1 - \mathbf{v}_2$ (since the dot product is zero). That is all that we get–there is no condition that $\mathbf{v}_1 - \mathbf{v}_2$ must equal zero.

Problem (12.3.48) Find the acute angle between the lines $y = \sqrt{3}x - 1$ and $y = -\sqrt{3}x + 2$.

Hints:
Line perpendicular to a vector in $\mathbb{R}^2$: The vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to the line $ax + by = c$.
Line parallel to a vector in $\mathbb{R}^2$: The vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is parallel to the line $bx - ay = c$.

We only need to use one of the hints. Let's use the one about parallel lines. Sketch of the situation:



Let $v_1 = \langle -1, \sqrt{3} \rangle$ and $v_2 = \langle 1, \sqrt{3} \rangle$. Then

$$\begin{aligned}
|v_1| = |v_2| &= 2 \\
v_1 \cdot v_2 &= -1 + 3 = 2 \\
\theta &= \arccos\left(\frac{v_1 \cdot v_2}{|v_1|\,|v_2|}\right) = \arccos\left(\frac{2}{4}\right) = \arccos\left(\frac{2}{4}\right) = \frac{\pi}{3}
\end{aligned}$$