Integral Formulas

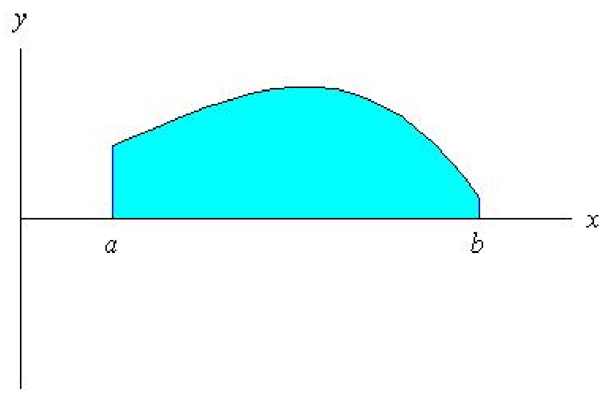
We learned a lot of integral formulas in this class. I've collected the most important ones here along with a little sketch to help you remember where they came from, and also to help you remember them.

You shouldn't try to memorize these–you should verify that you can reproduce them. The forms listed here are not meant to be exhaustive, simply representative.

For explicit functions, I am assuming $y = f(x)$, $a \le x \le b$.

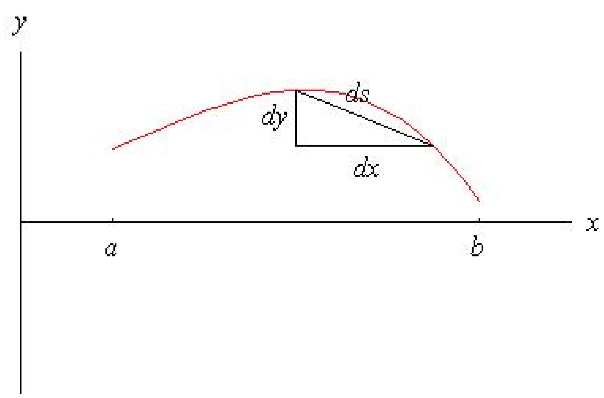
For parametric functions, I am assuming $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$. The endpoints satisfy $f(\alpha) = a$ and $f(\beta) = b$ from the explicit representation.

Area Under a Curve



Explicit: $A = \int_a^b f(x)\,dx$ Parametric: $A = \int_\alpha^\beta g(t)f'(t)\,dt$.

Arc Length

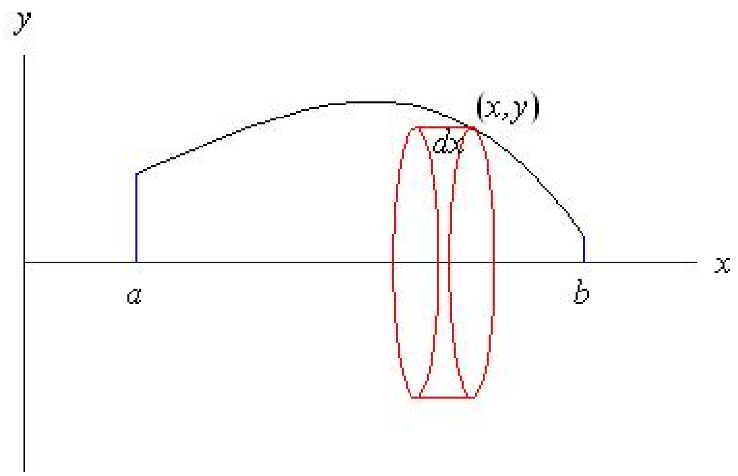


$(ds)^2 = (dx)^2 + (dy)^2$

Explicit: $L = \int_a^b ds, \quad ds = \sqrt{1 + f'(x)^2}\,dx = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$.

Parametric: $L = \int_\alpha^\beta ds, \quad ds = \sqrt{f'(t)^2 + g'(t)^2}\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

Volume of Revolution by Washers

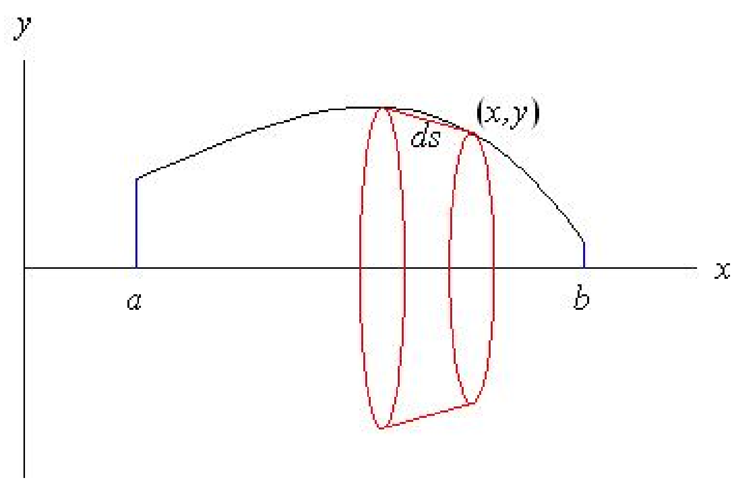


Note: The washer has side of length dx. The radius of the washer is y. The volume of the washer is $\pi y^2\,dx$. Adding up all the washers leads to the integral.

Explicit: $$V = \int_a^b \pi f(x)^2\,dx$$

Parametric: $$V = \int_\alpha^\beta \pi g(t)^2 f'(t)\,dt.$$

Surface Area of Revolution



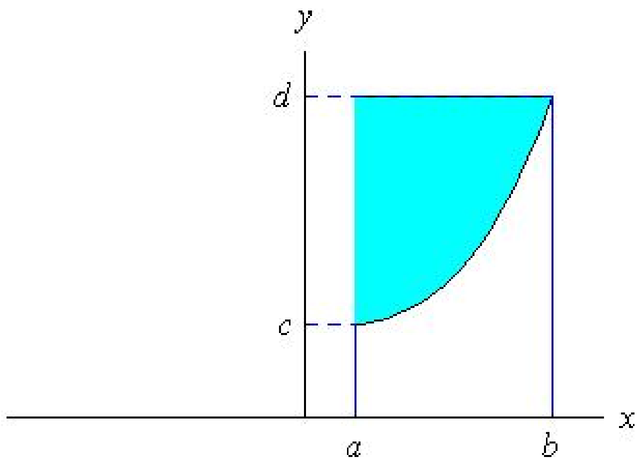
Note: The frustum has side of length ds. The radius of the circle is y. The surface area of the frustum (excluding circular faces) is approximately $2\pi y\,ds$. Adding up all the frustums leads to the integral.

Explicit: $$S = \int_a^b 2\pi f(x)\,ds, \quad ds = \sqrt{1+f'(x)^2}\,dx = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

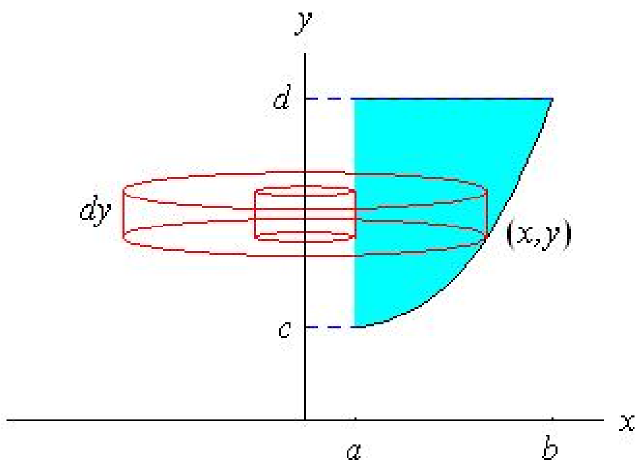
Parametric: $$S = \int_\alpha^\beta 2\pi g(t)\,ds, \quad ds = \sqrt{f'(t)^2+g'(t)^2}\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

Volume of Revolution about y-axis by Washers

This is a more complicated problem that shows how to build the integral for volume of revolution. We want to rotate the shaded region about the y-axis.



We construct washers, which will now have an inner and an outer radius.



Note: The washer has side of height dy. The inner radius of the washer is a. The outer radius of the washer is x. The volume of the washer is $\pi(x^2 - a^2)\,dy$. Adding up all the washers leads to the integral.

Explicit: $$V = \int_c^d \pi(x^2 - a^2)\,dy$$

We have to solve $y = f(x)$ for $x = g(y)$, and then we have $c = g(a)$, $d = g(b)$, and the volume of revolution is

Explicit: $$V = \int_c^d \pi(g^2(y) - a^2)\,dy$$

This example shows why we don't want to memorize formulas, since the formula can be different depending on what axis you rotate around.