Example: convergence Find the interval of convergence of $\sum_{n=1}^{\infty}(-1)^n \frac{(x+2)^n}{n2^n}$.

Here, we use the ratio test to determine the radius of convergence first. $a_n = (-1)^n \frac{(x+2)^n}{n2^n}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|(-1)^{n+1}\frac{(x+2)^{n+1}}{(n+1)2^{n+1}} \times (-1)^n \frac{n2^n}{(x+2)^n}\right| \\
&= \frac{1}{2}|x+2| \lim_{n\to\infty}\left|\frac{n}{(n+1)}\right| \\
&= \frac{1}{2}|x+2| \lim_{n\to\infty}\left|\frac{1}{(1+1/n)}\right| \\
&= \frac{1}{2}|x+2|
\end{aligned}$$

If this is less than 1, the series converges, so the series converges if $|x+2| < 2$.
This is the same as the interval $-4 < x < 0$, since the center is $a = -2$ and the radius of convergence is $R = 2$.

We need to check endpoints individually, since the ratio test tells us nothing there.

Consider $x = -4$: $\sum_{n=1}^{\infty}(-1)^n \frac{(-4+2)^n}{n2^n} = \sum_{n=1}^{\infty}(-1)^n \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$

This is the divergent harmonic series.

Consider $x = 0$: $\sum_{n=1}^{\infty}(-1)^n \frac{(+2)^n}{n2^n} = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$

This is an alternating series, with $b_n = \frac{1}{n}$. It is convergent since $b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$ and $\lim_{n\to\infty} b_n = 0$.

So the interval of convergence for $\sum_{n=1}^{\infty}(-1)^n \frac{(x+2)^n}{n2^n}$ is $-4 < x \leq 1$.

Example: Taylor Series Find the Taylor series of $f(x) = \ln x$ about $x = 1/2$. What is the radius of convergence?

You could use a table and look for a pattern to answer this question, but instead I am going to make this look like a geometric series.

$$\begin{aligned}
\frac{d}{dx}\ln x &= \frac{1}{x} \\
&= \frac{1}{1/2 - (1/2 - x)} \\
&= \frac{2}{1 - (1 - 2x)} \\
&= 2\sum_{n=0}^{\infty}(1-2x)^n, \quad |1-2x| < 1 \quad \text{using geometric series result}
\end{aligned}$$

$$
\begin{aligned}
\ln x &= 2\sum_{n=0}^{\infty}\int (1-2x)^n\,dx, \quad |x-1/2|<1/2 \\
&\quad \text{Substitution: } w = 1-2x,\ dw = -2\,dx \\
&= -\sum_{n=0}^{\infty}\int w^n\,dw, \quad |x-1/2|<1/2 \\
&= -\sum_{n=0}^{\infty}\frac{w^{n+1}}{n+1} + c, \quad |x-1/2|<1/2 \\
&= -\sum_{n=0}^{\infty}\frac{(1-2x)^{n+1}}{n+1} + c, \quad |x-1/2|<1/2
\end{aligned}
$$

If we evaluate this at $x = 1/2$ we can determine the value of c.

$$
\begin{aligned}
\ln(1/2) &= +c \\
\ln x &= -\sum_{n=0}^{\infty}\frac{(1-2x)^{n+1}}{n+1} + \ln(1/2), \quad |x-1/2|<1/2 \\
&= -\ln 2 - \sum_{n=0}^{\infty}\frac{(1-2x)^{n+1}}{n+1}, \quad |x-1/2|<1/2
\end{aligned}
$$

The radius of convergence is $1/2$.

Example: Integral test Show the following series is divergent using the integral test. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$.

The integral test requires that we work with $f(x)$, where
1) $f(n) = a_n$,
and on the interval $[1,\infty)$, $f(x)$ is:
1) continuous,
2) positive,
3) decreasing.

Here, $f(x) = \dfrac{\ln x}{x}$, which is continuous and positive on the interval $[1,\infty)$.

But is it decreasing on this interval? It is not obvious, since both the numerator and denominator are increasing functions of x.

However, if a function $f(x)$ is decreasing, then it must be true that $f'(x) < 0$. Let's take the derivative of $f(x)$ and see what we can learn.

$$
\frac{d}{dx}f(x) = \frac{d}{dx}\frac{\ln x}{x} = \frac{1-\ln x}{x^2}
$$

For this to be less than zero, we require $1 - \ln x < 0 \longrightarrow x > e$. This will certainly be true if $x > 3$, since $e \sim 2.71828$.

We can therefore apply the integral test to the series $\displaystyle\sum_{n=3}^{\infty}\frac{\ln n}{n}$. Note that we start at $n = 3$ and not $n = 1$, since we must work on the interval $[3, \infty)$.

$$\begin{aligned}
\int_3^\infty f(x)\,dx &= \int_3^\infty \frac{\ln x}{x}\,dx \\
&= \lim_{t\to\infty} \int_3^t \frac{\ln x}{x}\,dx \\
&\quad \text{Substitution: } \begin{array}{ll} u = \ln x & \text{when } x = 3, u = \ln 3 \\ du = \frac{1}{x}dx & \text{when } x = t, u = \ln t \end{array} \\
&= \lim_{t\to\infty} \int_{\ln 3}^{\ln t} u\,du \\
&= \frac{1}{2} \lim_{t\to\infty} u^2\Big|_{\ln 3}^{\ln t} \\
&= \frac{1}{2} \lim_{t\to\infty} \left(\ln^2 t - \ln^2 3\right) \\
&= \infty, \quad \text{diverges, since } \ln^2 t \to \infty \text{ as } t \to \infty.
\end{aligned}$$

Since the integral diverges, the series $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges by the integral test. Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.