The techniques of integration are important, and can show up in other types of problems. For example, maybe to calculate the surface area of a solid you need to do an integral that requires you to use some trig identities. And to get the mean value of a probability distribution frequently requires the use of parts.

Example: Trig Substitution Evaluate the integral using the substitution $x = 2\sec\theta$.

$$\int \frac{(x^2-4)^{3/2}}{x^6}\,dx.$$

Although this doesn't have an obvious square root, which is what usually tips us off to try a trig substitution, we have been told that a trig substitution will work. So let's proceed.

$$\begin{aligned} &x = 2\sec\theta \\ &dx = 2\sec\theta\tan\theta\,d\theta \\ &\text{where } 0 < \theta < \frac{\pi}{2} \text{ or } \pi < \theta < \frac{3\pi}{2} \end{aligned}$$

Now, we find expressions for the components of the integrand:

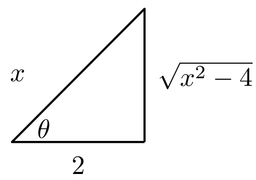
$$\begin{aligned} (x^2-4)^{3/2} &= (4\sec^2\theta - 4)^{3/2} \\ &= 2^3(\sec^2\theta - 1)^{3/2} \\ &= 2^3(\tan^2\theta)^{3/2} \\ &= 2^3|\tan\theta|^3 \\ &= 2^3\tan^3\theta \text{ (since } \tan\theta > 0 \text{ in our restricted domain for } \theta!) \\ \\ x^6 &= 2^6\sec^6\theta \end{aligned}$$

And now we do the integral:

$$\begin{aligned} \int \frac{(x^2-4)^{3/2}}{x^6}\,dx &= \int \frac{(2^3\tan^3\theta)}{(2^6\sec^6\theta)}(2\sec\theta\tan\theta\,d\theta) \\ &= \frac{1}{2^2}\int \frac{\tan^4\theta}{\sec^5\theta}\,d\theta \\ &= \frac{1}{4}\int \frac{\sin^4\theta}{\cos^4\theta}\cos^5\theta\,d\theta \\ &= \frac{1}{4}\int \sin^4\theta\cos\theta\,d\theta \quad \text{Substitution: } \begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array} \\ &= \frac{1}{4}\int u^4\,du \\ &= \frac{1}{4}\left(\frac{u^5}{5}\right) + C \\ &= \frac{1}{20}\sin^5\theta + C \end{aligned}$$

We now need to back substitute for θ. Construct the diagram that will help us back substitute the θ:

$$\sec\theta = \frac{x}{2} \longrightarrow \cos\theta = \frac{2}{x}$$



$$\sin\theta = \frac{\sqrt{x^2-4}}{x}$$

$$\int \frac{(x^2-4)^{3/2}}{x^6}\,dx = \frac{1}{20}\left(\frac{\sqrt{x^2-4}}{x}\right)^5 + C = \frac{(x^2-4)^{5/2}}{20x^5} + C.$$

Example: Separable Differential Equation Solve the initial value problem given below for $y(x)$.

$$x + 2y\sqrt{x^2+1}\,\frac{dy}{dx} = 0, \quad y(0) = 1.$$

$$\begin{aligned}
x + 2y\sqrt{x^2+1}\,\frac{dy}{dx} &= 0 \\
2y\,dy &= -\frac{x}{\sqrt{x^2+1}}\,dx \quad \text{(separate)} \\
\int 2y\,dy &= -\int \frac{x}{\sqrt{x^2+1}}\,dx \quad \text{(integrate)} \\
y^2 + c_1 &= -\int \frac{x}{\sqrt{x^2+1}}\,dx \quad \text{Substitute: } \begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array} \\
y^2 + c_1 &= -\int \frac{1}{2\sqrt{u}}\,du \\
y^2 + c_1 &= -\frac{1}{2}\int u^{-1/2}\,du \\
y^2 + c_1 &= -\frac{1}{2}\frac{u^{1/2}}{1/2} + c_2 \\
y^2 &= -\sqrt{x^2+1} + c, \quad c = c_2 - c_1
\end{aligned}$$

The above is an implicit solution to the differential equation. It is a family of curves.

Now we must use the initial condition to determine the constant c. This will pick the curve that passes through the point $(x_0, y_0) = (0, 1)$ out from the family of curves.

$$\begin{aligned}
1^2 &= -\sqrt{0^2+1} + c \\
c &= 2
\end{aligned}$$

So the solution to the initial value problem is

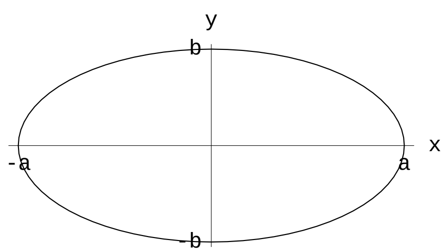
$$y^2 = -\sqrt{x^2+1} + 2.$$

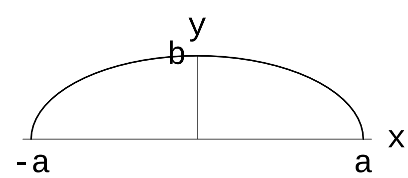
You can check that this is correct by taking an implicit derivative.

Example: Arc Length Set up, but do not evaluate, an integral for the length of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a, b \text{ constants.}$$

The function is an ellipse, which I have sketched below.





We can work with the top half of the ellipse, and get 1/2 of the arc length. If we do this, we can work with the explicit function

$$y = b\sqrt{1 - \frac{x^2}{a^2}} = b\left(1 - \frac{x^2}{a^2}\right)^{1/2}.$$

$$\begin{aligned}
\text{Arc Length} &= \int ds \\
ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
\frac{dy}{dx} &= b\left(\frac{1}{2}\right)\left(1 - \frac{x^2}{a^2}\right)^{-1/2}\left(-2\frac{x}{a^2}\right) \\
&= \frac{-bx/a^2}{\sqrt{1 - x^2/a^2}} \\
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{b^2x^2/a^4}{1 - x^2/a^2} \\
\text{Arc Length of ellipse} &= 2\int_{-a}^{a} \sqrt{1 + \frac{b^2x^2/a^4}{1 - x^2/a^2}}\; dx
\end{aligned}$$

This solution is fine, but cumbersome. For example, to check if the solution is correct you may want to verify that it gives the circumference of a circle of radius r $(a = b = r)$ to be $2\pi r$. You would need to do a trig substitution to verify this, so it is not the sort of check you would probably do. In fact, there is no nice way of checking this.

Another solution would be to work with the parametric expression for the ellipse:

$$
\begin{aligned}
x &= a\cos t \\
y &= b\sin t \\
& 0 \le t \le 2\pi
\end{aligned}
$$

This will sweep out the entire ellipse.

$$
\begin{aligned}
\text{Arc Length} &= \int ds \\
ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
\frac{dx}{dt} &= -a\sin t \\
\frac{dy}{dt} &= b\cos t \\
\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{a^2 sin^2 t + b^2\cos^2 t} \\
\text{Arc Length of ellipse} &= \int_0^{2\pi} \sqrt{a^2 sin^2 t + b^2\cos^2 t}\ dt
\end{aligned}
$$

From this representation it is very easy to see that the circumference of a circle of radius r ($a = b = r$) is $2\pi r$.