Leibniz Notation

Leibniz notation of derivatives is a powerful and useful notation that makes the process of computing derivatives clearer than the prime notation. So, what is Leibniz notation?

For $y = f(x)$, the derivative can be expressed using prime notation as

$$y', f'(x),$$

or using Leibniz notation as

$$\frac{dy}{dx}, \frac{d}{dx}[y], \frac{df}{dx}, \frac{d}{dx}[f(x)].$$

The advantage of the Leibniz notation over the prime notation is that it makes explicitly clear what you are differentiating with respect to:

$\frac{dy}{dx}$ is read as "the derivative of y with respect to x".

$\frac{d}{dx}[y]$ is read as "the derivative with respect to x of y".

Of course, since these quantities are the same thing you can swap how those are read.

Once we start doing more complicated derivatives like the chain rule or implicit differentiation, the Leibniz notation shows its worth.

Chain Rule with Leibniz Notation

If a function is defined by a composition $y = f(g(x))$, it can be decomposed as $y = f(u), \quad u = g(x)$.
The derivative of y with respect to x is then computed using the chain rule as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Using Leibniz notation easily allows one to easily create longer chains when there is more nesting in the composition.
If $y = f(g(h(x)))$ decompose as: $y = f(u), \quad u = g(w), \quad w = h(x)$, and the chain rule now has two links in the chain:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx}$$

Example $y = \tan(\cos(x^2))$, find dy/dx.

Decompose: $y = \tan u, \quad u = \cos w, \quad w = x^2$

Apply chain rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx} \\
&= \frac{d}{du}[y] \cdot \frac{d}{dw}[u] \cdot \frac{d}{dx}[w] \text{ (this step optional; I am showing it here to clarify the Leibniz notation)} \\
&= \frac{d}{du}[\tan u] \cdot \frac{d}{dw}[\cos w] \cdot \frac{d}{dx}[x^2] \\
&= [\sec^2 u][-\sin w][2x] \\
&= -2x \sec^2(\cos x^2) \sin(x^2)
\end{aligned}$$

Example Find the derivative of the function $y = \sin(\tan\sqrt{\sin x})$.

Note: If you can do the decomposition in your head, you don't need to show it in your solution. This solution just shows how Leibniz notation can keep your solution organized if you need it.

We will need multiple applications of the chain rule to do this derivative. Let's do that first before we take any derivatives.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}[y] = \frac{d}{dx}[\sin(\tan\sqrt{\sin x})] \\
&= \frac{d}{dx}[\sin u], && u = \tan\sqrt{\sin x} \\
&= \frac{d}{du}[\sin u]\cdot\frac{du}{dx}, && \text{(chain rule)} \\
& && u = \tan v, v = \sqrt{\sin x} \\
&= \frac{d}{du}[\sin u]\cdot\frac{du}{dv}\cdot\frac{dv}{dx}, && \text{(chain rule a second time)} \\
& && u = \tan v, v = \sqrt{w}, w = \sin x \\
&= \frac{d}{du}[\sin u]\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx}, && \text{(chain rule a third time)}
\end{aligned}$$

All this was just setting up the derivative in a manner that we could find it by using multiple applications of the chain rule. Now we can take the derivatives.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{du}[\sin u]\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx} \\
&= \frac{d}{du}[\sin u]\cdot\frac{d}{dv}[\tan v]\cdot\frac{d}{dw}[w^{1/2}]\cdot\frac{d}{dx}[\sin x] \\
&= (\cos u)\cdot(\sec^2 v)\cdot\left(\frac{1}{2}w^{-1/2}\right)\cdot(\cos x) \\
&= (\cos u)\cdot(\sec^2 v)\cdot\left(\frac{1}{2\sqrt{w}}\right)\cdot(\cos x) \\
&= (\cos(\tan\sqrt{\sin x}))\cdot(\sec^2\sqrt{\sin x})\cdot\left(\frac{1}{2\sqrt{\sin x}}\right)\cdot(\cos x) \\
&= \frac{\cos x\cos(\tan\sqrt{\sin x})\sec^2\sqrt{\sin x}}{2\sqrt{\sin x}}
\end{aligned}$$

Using Leibniz Notation with Implicit Differentiation

Example Find the slope of the curve $x^4 + y^2 - 25 = 0$ at the point (x, y).

The slope of the curve is the derivative of the curve, so we want to find dy/dx.

$$\begin{aligned}
& x^4 + y^2 - 25 = 0 \quad \text{the implicit function} \\
& \frac{d}{dx}[x^4 + y^2 - 25 = 0] \quad \text{differentiate the equation} \\
& \frac{d}{dx}[x^4] + \frac{d}{dx}[y^2] - \frac{d}{dx}[25] = \frac{d}{dx}[0] \quad \text{sum and difference rules} \\
& 4x^3 + \frac{d}{dy}[y^2]\cdot\frac{dy}{dx} - 0 = 0 \quad \text{power rule, chain rule, constant rule} \\
& 4x^3 + 2y\frac{dy}{dx} = 0 \\
& \frac{dy}{dx} = -\frac{2x^3}{y}
\end{aligned}$$

This is valid for points on the top or bottom half of the curve.

Using Leibniz Notation to Keep Track of Pieces of a Long Derivative

Leibniz notation is also exceptionally good as keeping track of what is happening during the derivative of a complicated function (one that involves a combination of product rule, quotient rule, chain rule, etc).

Example Differentiate $\dfrac{e^{x^2}}{1+x}$.

$$\begin{aligned}
y &= \frac{e^{x}}{1+x} \\
\frac{dy}{dx} &= \frac{d}{dx}\left[\frac{e^{x^2}}{1+x}\right] \\
&= \frac{(1+x)\frac{d}{dx}[e^{x^2}] - e^{x^2}\frac{d}{dx}[1+x]}{(1+x)^2} \text{ (quotient rule)} \\
&= \frac{(1+x)\cdot 2x \cdot e^{x^2} - e^{x^2}\cdot 1}{(1+x)^2} \text{ (chain rule)} \\
&= \frac{(2x^2+2x-1)e^{x^2}}{(1+x)^2} \text{ (simplify)}
\end{aligned}$$

Example Find $f'(x)$ given $f(x) = \dfrac{\sqrt{x}}{x-1} - 22e^x + \sec x \sin x$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\frac{\sqrt{x}}{x-1} - 22e^x + \sec x \sin x\right] \\
&= \frac{d}{dx}\left[\frac{\sqrt{x}}{x-1}\right] - 22\frac{d}{dx}\left[e^x\right] + \frac{d}{dx}\left[\sec x \sin x\right] \quad \text{difference rule, constant multiple rule} \\
&= \frac{(x-1)\dfrac{d}{dx}[x^{1/2}] - x^{1/2}\dfrac{d}{dx}[x-1]}{(x-1)^2} - 22e^x + \sec x\frac{d}{dx}\left[\sin x\right] + \sin x\frac{d}{dx}\left[\sec x\right] \quad \text{product, quotient rule} \\
&= \frac{(x-1)[\frac{1}{2}x^{-1/2}] - x^{1/2}[1]}{(x-1)^2} - 22e^x + \sec x(\cos x) + \sin x(\sec x \tan x) \quad \text{power rule, exponential rule} \\
&= \frac{\frac{1}{2}\sqrt{x} - \frac{1}{2\sqrt{x}} - \sqrt{x}}{(x-1)^2} - 22e^x + 1 + \tan^2 x \quad \text{simplify} \\
&= \frac{-\frac{1}{2}\sqrt{x} - \frac{1}{2\sqrt{x}}}{(x-1)^2} - 22e^x + 1 + \tan^2 x \quad \text{simplify} \\
&= \frac{-x-1}{2\sqrt{x}(x-1)^2} - 22e^x + \sec^2 x \quad \text{simplify}
\end{aligned}$$