1102 Calculus II 8.4 Economics

Marginal Cost

Marginal Cost is the instantaneous rate of change of the cost with respect to the number of a commodity produced (c.f. Sec 3.3 and 4.8).

Notation:
x: number of commodity produced. Units: [item]
$C(x)$: cost of producing x units of the commodity. Units: [\$]
$M(x)$: marginal cost. Units: [\$/item]

Relation:

$$\begin{aligned} M(x) &= \frac{d}{dx}C(x) \\ M(x)\,dx &= dC(x) \\ \int M(x)\,dx &= \int dC(x) = C(x) \\ C(x) &= \int M(x)\,dx \end{aligned}$$

Consumer Surplus

The demand function $p(x)$ is the price a company has to charge in order to sell x units of a commodity.

For example, if you are selling hamburgers, you may find that the following is true:

Price	Amount People are Willing to Buy
\$4.00	1
\$2.50	2
\$2.00	3
\$1.50	4
\$1.00	5
\$0.50	6

Some people may buy more than one hamburger, and that is included in the table. This table represents the demand function.

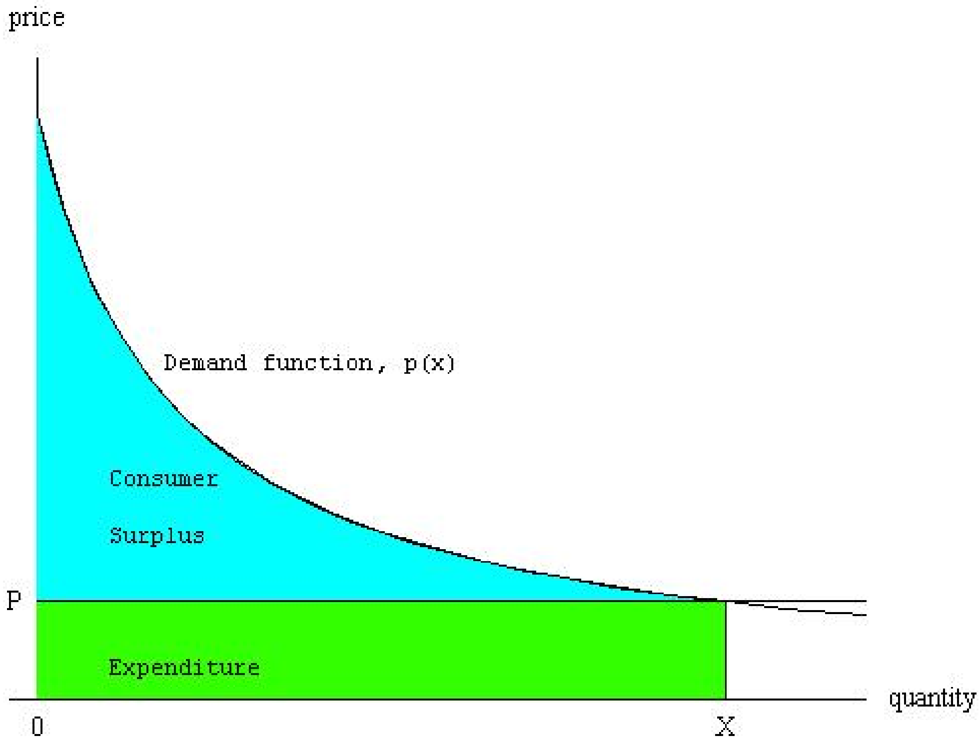
In our example, if you want to sell four hamburgers, you need to sell them for \$1.50.

The four hamburgers you sold would cost consumers $4 \times \$1.50 = \$6.00$.

However, some of the people who bought your hamburgers would have been willing to spend more. The value of those four hamburgers to the consumers is $1 \times \$4.00 + 2 \times \$2.50 + 1 \times \$2.00 = \$11.00$ for the four hamburgers. The consumer surplus is $\$11.00 - \$6.00 = \$5.00$.

The consumer surplus can be thought of the difference between the *value in use* and the *value in exchange*.

Graphically, this can be represented as an area under a curve, where instead of the table of discrete values we used above the demand function is continuous.



P is the current selling price, in dollars, of the commodity.
X is the amount of the commodity currently being sold.
The consumer expenditure (green shaded area in diagram) represents the amount of money the consumers spent on purchasing X units of the commodity.

$$\text{Consumer Expenditure} = \int_0^X P\,dx = XP.$$

The consumer surplus (blue shaded area in diagram) represents the amount of money saved by consumers in purchasing the commodity at price P.

$$\text{Consumer Surplus} = \int_0^X \Big(p(x) - P\Big)\,dx.$$

The total value to consumers is the consumer surplus added to the expenditure

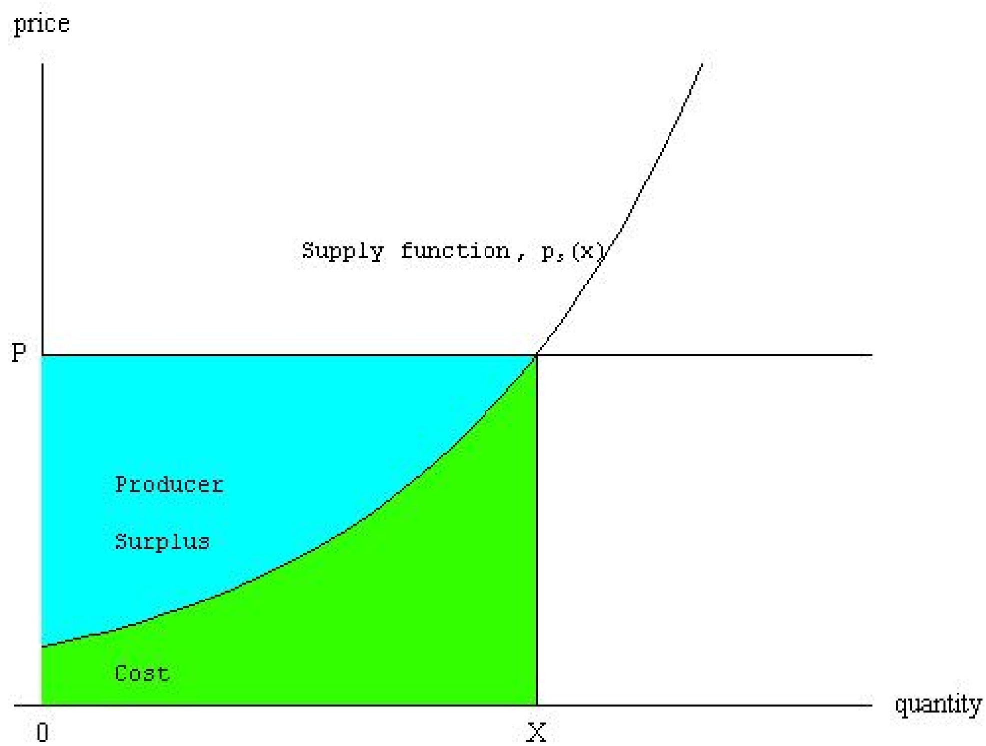
$$\text{Value to Consumers} = \int_0^X p(x)\,dx.$$

Producer Surplus

Similar to the consumer surplus, you can look at the manufacturing of goods from the point of view of the manufacturer, and define the producer surplus.

The supply function $p_s(x)$ gives the relation between the the number of units that the manufacturer produces and the price to produce them. If the price is increased, the number of units created will increase.

Graphically, this can be represented as an area under a curve.



P is the current selling price, in dollars, of the commodity.
X is the amount of the commodity currently being sold.
The manufacturing cost (green shaded area in diagram) represents the amount of money the manufacturer spent to make the product.

$$\text{Manufacturing Cost} = \int_0^X p_s(x)\,dx.$$

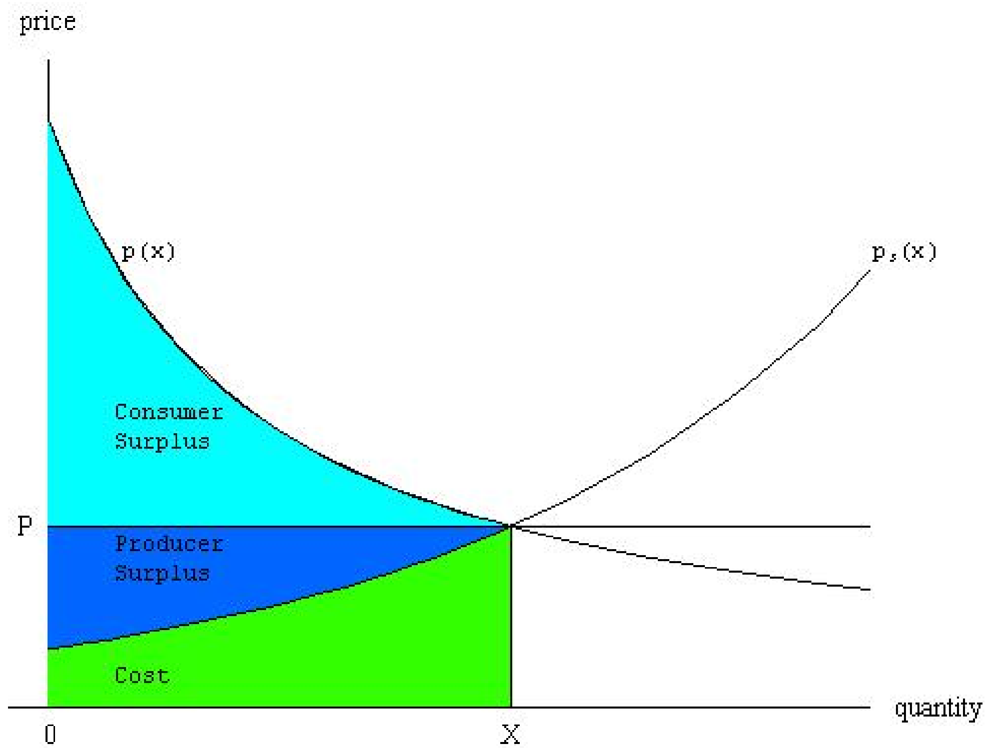
The producer surplus (net income) (blue shaded area in diagram) represents the amount of money made by the manufacturer by selling the commodity at price P.

$$\text{Producer Surplus} = \int_0^X \Big(P - p_s(x)\Big)\,dx.$$

The gross income is the same as the consumer expenditure:

$$\text{Gross Income} = \text{ Consumer Expenditure } = \int_0^X P\,dx = XP.$$

The most efficient level of production results in the highest total surplus, that is, the total of consumer and producer surplus is maximized. This is sketched below.



To understand why this is the most efficient level of production so goes beyond this course, but you could look for discussions regarding consumer/producer surplus in microeconomic texts if you are interested in this topic.

Example The marginal cost of installing telephones in an office building is

$$M(t) = 74 + 1.1t - 0.002t^2 \text{ [\$/telephone]}$$

where t is the number of telephones installed.

The firm installing the telephones determined the marginal cost equation by examining data it has from its years of experience of installing phones.

Find the increase in cost if the building management decides to install 1600 phones instead of 1200.

Solution We need to determine the cost of installing t telephones in the building. This will be $C(t)$.

$$\begin{aligned} C(t) &= \int M(t)\,dt \\ &= \int (74 + 1.1t - 0.002t^2)\,dt \\ &= 74t + 0.55t^2 - \frac{0.002}{3}t^3 + c_1 \text{ [\$]} \end{aligned}$$

The increase in cost of installing 1600 phones instead of 1200 phones is

$$C(1600) - C(1200) = \$933\ 067.$$

The value $C(0) = c_1$ is the fixed startup cost.

Example 8.4.10 A movie theater has been charging $7.50 per person and selling about 400 tickets on a typical weeknight. After surveying their customers, the theater estimates that for every $0.50 that they lower their price, the number of moviegoers will increase by 35 per night. Find the demand function and calculate the consumer surplus when the tickets are priced at $6.00.

Solution We need a table of values to help us get the demand curve.

Ticket Price = $p(x)$ [$]	Number of patrons = x
7.5	400
7.0	435
6.5	470
6.0	505

We actually have a linear relation here, so we can get the equation of the demand curve using the two point equation of a straight line:

$$\frac{x - x_0}{y - y_0} = \frac{x_1 - x_0}{y_1 - y_0}$$

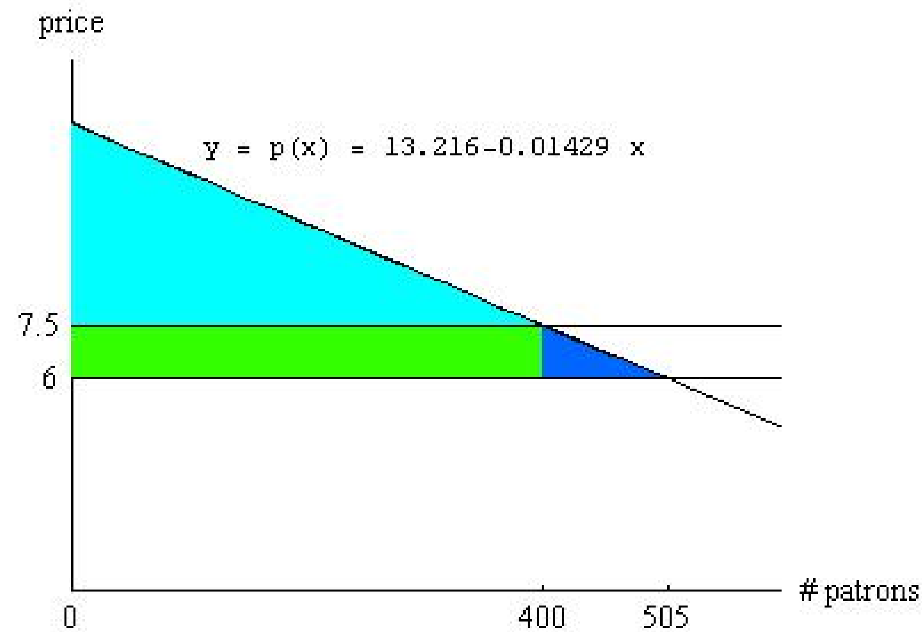
will give the linear curve which passes through the points (x_0, y_0), (x_1, y_1).

We can pick any two points from the table; let's choose $(7.5, 400)$ and $(6.5, 470)$.

$$\begin{aligned} \frac{x - x_0}{y - y_0} &= \frac{x_1 - x_0}{y_1 - y_0} \\ \frac{x - 400}{y - 7.5} &= \frac{470 - 400}{6.5 - 7.5} = -70 \\ y &= -\frac{1}{70}x + \frac{400}{70} + 7.5 \\ y &= 13.216 - 0.01429x \end{aligned}$$

The demand curve is $p(x) = 13.216 - 0.01429x$.

The consumer surplus when tickets cost \$6 is all the shaded regions (dark blue, light blue, and green) in the following sketch.



The light blue region represents the consumer surplus if the ticket price is left at \$7.50.

$$\begin{aligned}
\text{Consumer Surplus if tickets cost \$7.50} &= \int_0^{400} (p(x) - 7.5)\, dx \\
&= \int_0^{400} (5.716 - 0.01429x)\, dx \\
&= \left(5.716x - \frac{0.01429}{2} x^2 \right)_0^{400} \\
&= \$1143.30
\end{aligned}$$

The green region represents the increase in the consumer surplus in lowering ticket prices from \$7.50 to \$6.00 that comes from the continuing 400 patrons each night.

$$\begin{aligned}
\text{Increase in Consumer Surplus due to old customers} &= \int_0^{400} (7.5 - 6)\, dx \\
&= 400(1.5) = \$600.00
\end{aligned}$$

The dark blue region represents the increase in the consumer surplus in lowering ticket prices that comes from the 105 new patrons each night.

$$\begin{aligned}
\text{Increase in Consumer Surplus due to new customers} &= \int_{400}^{505} (p(x) - 6)\, dx \\
&= \int_{400}^{505} (7.216 - 0.01429x)\, dx \\
&= \left(7.216x - \frac{0.01429}{2}x^2\right)_{400}^{505} \\
&= \$78.73
\end{aligned}$$

The sum of all the shaded regions is the consumer surplus when tickets cost \$6.00. From our calculation above, this should be $1143.20 + 600.00 + 78.73 = \$1821.93$.

We could, of course, calculate this directly:

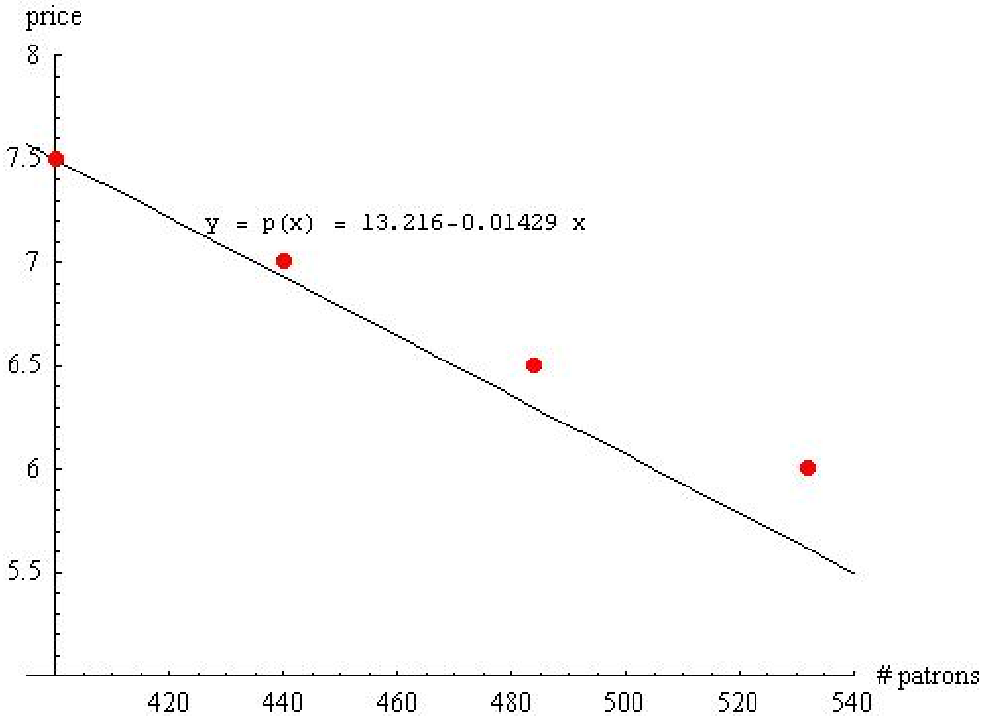
$$\begin{aligned}
\text{Consumer Surplus if tickets cost \$6.00} &= \int_{0}^{505} (p(x) - 6.00)\, dx \\
&= \int_{0}^{505} (7.216 - 0.01429x)\, dx \\
&= \left(7.216x - \frac{0.01429}{2}x^2\right)_{0}^{505} \\
&= \$1821.93
\end{aligned}$$

Another interesting question to ask is what should the theater charge to maximize profits? This will lead to a maximization problem you've seen in Calculus I. Solution: \$6.61, assuming profit $= p(x)x$. This assumes all profits come from ticket sales, and there is no cost with having more people in the theater–a highly unlikely scenario!

Aside: What if the number of people who bought tickets increased by 10%, rather than 35 people, for every \$0.50 decrease in price?

What changes is the demand curve–it will no longer be linear. This drastically changes the problem!

Ticket Price $= p(x)$ [\$]	Number of patrons $= x$
7.5	400
7.0	440
6.5	484
6.0	532


price
8
7.5
7
6.5
6
5.5
y = p(x) = 13.216-0.01429 x
420
440
460
480
500
520
540
patrons