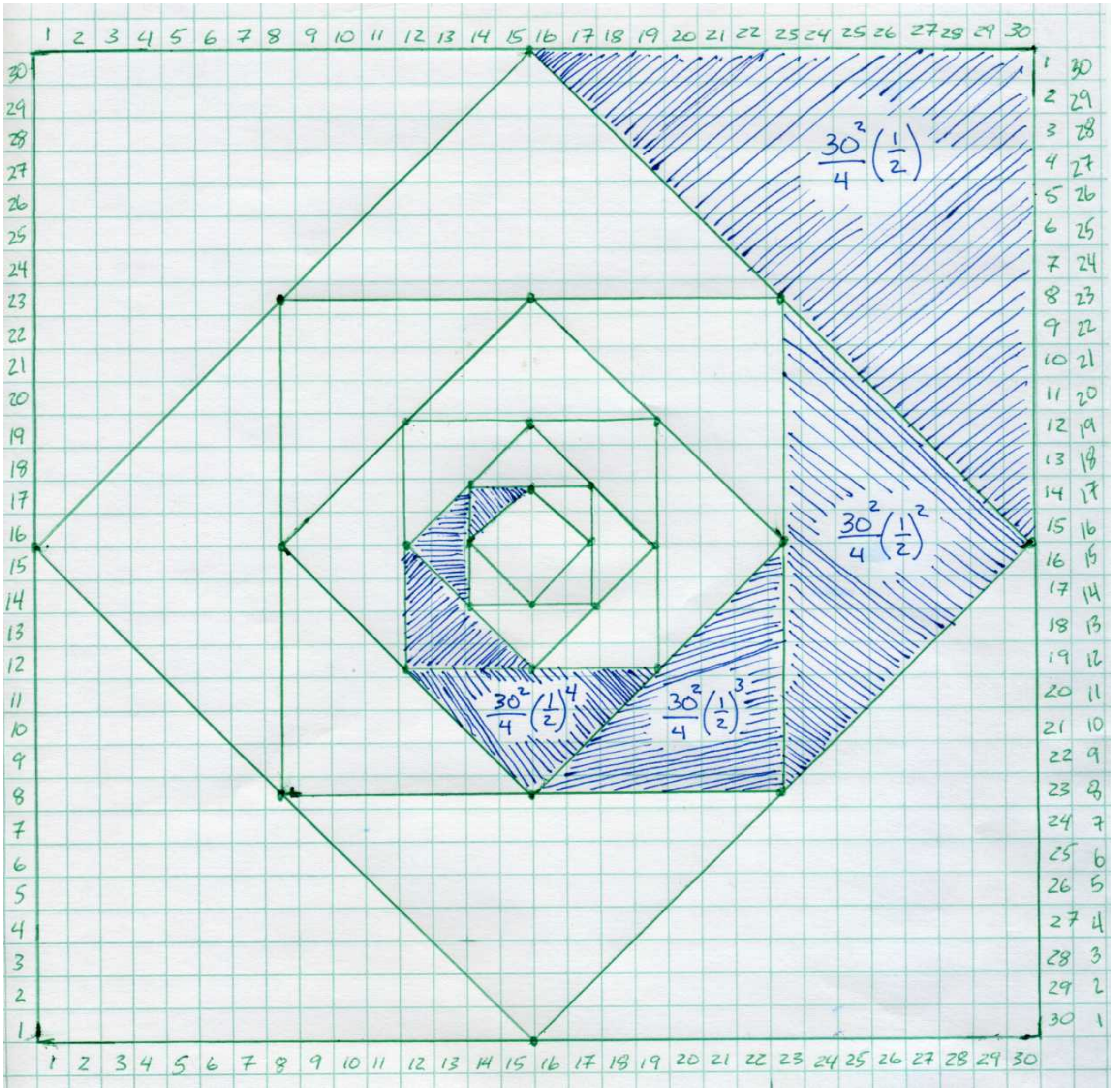
Consider the following diagram, which shows squares inscribed inside of themselves. The process can continue on to an infinite number of squares, but of course I can not draw that! (Notice I screwed up the numbering on the right–the numbers should increase going up if the bottom left is the origin)



From the diagram, we can see both a sequence and a series.

• The sequence is the size of the shaded triangles, and is given by

$$\left\{\frac{30^2}{4}\left(\frac{1}{2}\right)^n\right\}_{n=1}^{\infty} = \left\{\frac{30^2}{4}\left(\frac{1}{2}\right)^1, \frac{30^2}{4}\left(\frac{1}{2}\right)^2, \frac{30^2}{4}\left(\frac{1}{2}\right)^3, \frac{30^2}{4}\left(\frac{1}{2}\right)^4, \frac{30^2}{4}\left(\frac{1}{2}\right)^5, \ldots, \frac{30^2}{4}\left(\frac{1}{2}\right)^n, \ldots\right\}$$

So $a_n = \frac{30^2}{4}\left(\frac{1}{2}\right)^n$. The limit of the sequence is $\lim_{n\to\infty} a_n = 0$ since the 1/2 raised to an infinite power will be zero. This limit represents the fact that the size of the shaded triangles is approaching zero.

• If we add up all the shaded areas, we get a series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{30^2}{4}\left(\frac{1}{2}\right)^n.$$

This is actually a geometric series, so we can get its sum:

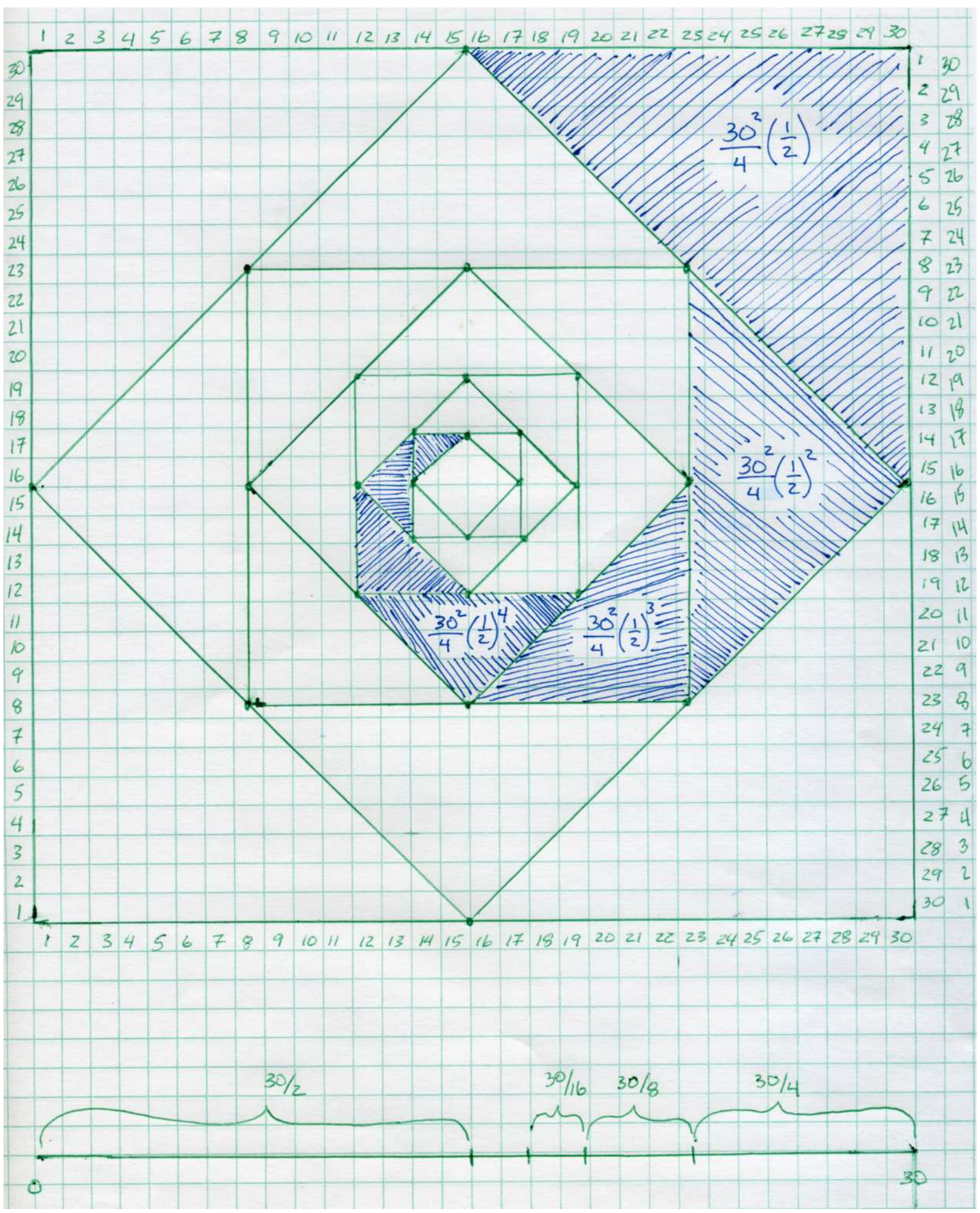
$$\sum_{n=1}^{\infty} \frac{30^2}{4} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{30^2}{8} \left(\frac{1}{2}\right)^{n-1} = \frac{30^2}{8} \times \frac{1}{1-1/2} = \frac{30^2}{4}.$$

The above equation implies the following:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{30^2}{4} \left(\frac{1}{2}\right)^n &= \frac{30^2}{4} \\ \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n &= 1 \end{aligned}$$

which is a pretty sweet result.

We can recover this result another way by looking at the x-axis in our diagram:



• From here, we can see that the sequence for the lengths along the x-axis is given by

$$\left\{30\left(\frac{1}{2}\right)^n\right\}_{n=1}^{\infty} = \left\{30\left(\frac{1}{2}\right)^1, 30\left(\frac{1}{2}\right)^2, 30\left(\frac{1}{2}\right)^3, 30\left(\frac{1}{2}\right)^4, 30\left(\frac{1}{2}\right)^5, \ldots, 30\left(\frac{1}{2}\right)^n, \ldots\right\}$$

So $a_n = 30\left(\frac{1}{2}\right)^n$. The limit of the sequence is $\lim_{n\to\infty} a_n = 0$ since the 1/2 raised to an infinite power will be zero. This limit represents the fact that the length of the interval is approaching zero.

• If we add up all the shaded areas, we get a series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 30\left(\frac{1}{2}\right)^n.$$

This is actually a geometric series, so we can get its sum:

$$\sum_{n=1}^{\infty} 30\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{30}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{30}{2} \times \frac{1}{1-1/2} = 30.$$

The above equation implies the following:

$$\begin{aligned} \sum_{n=1}^{\infty} 30\left(\frac{1}{2}\right)^n &= 30 \\ \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n &= 1 \end{aligned}$$

which is the same result we found earlier.

Things to note:

- Although the pictures provide good evidence that the series can be summed, they really only provide an upper bound for the sum:

$$\sum_{n=1}^{\infty} \frac{30^2}{4}\left(\frac{1}{2}\right)^n \leq \frac{30^2}{4}$$

$$\sum_{n=1}^{\infty} 30\left(\frac{1}{2}\right)^n \leq 30$$

  Using the geometric series result $\sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1-r}$ when $|r| < 1$ means we really used the sequence of partial sums to get the sum–since that was how we got the geometric series formula in the first place!
- This analysis did not require any calculus (in the sense of derivatives or integrals) to perform. However, it does require the understanding of infinity–which is at the heart of both differential and integral calculus. It is the dealing with infinity that connects all these topics.