Arc Length

The length of a curve in space (or *arc length*) can be a tricky concept to figure out at first. Here we shall take a closer look at how to correctly get an integral formula for arc length, and also how incorrect reasoning can lead us astray. What we learn will help us understand why we write the surface area of a surface of revolution as $\int 2\pi y\, ds$ instead of $\int 2\pi y\, dx$.

The Correct Formulation of Arc Length

Let's work with a specific example. Let's try to find the length of the curve $y = x^3 - 3x^2 - 4x + 2, \;\; -3 \leq x \leq 3$.

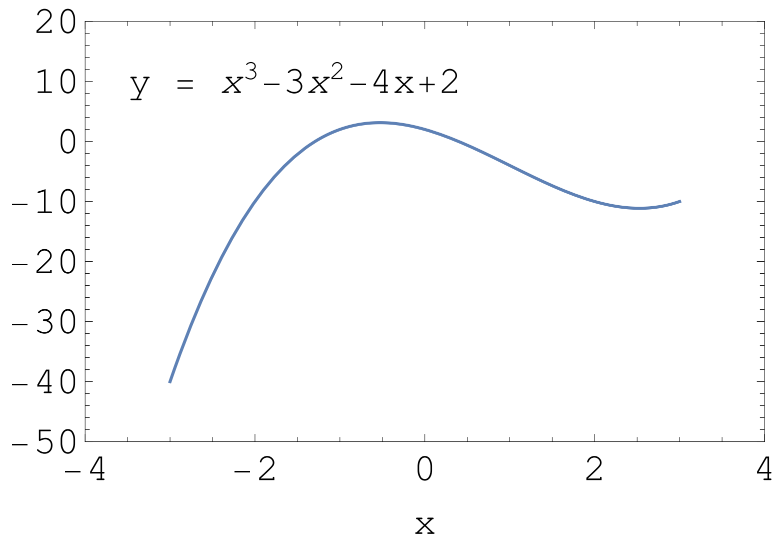
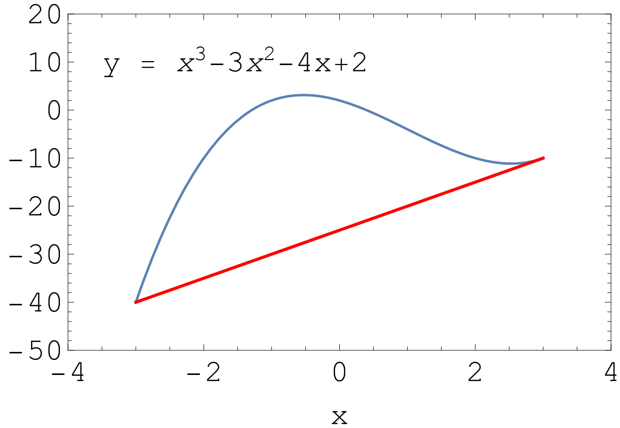


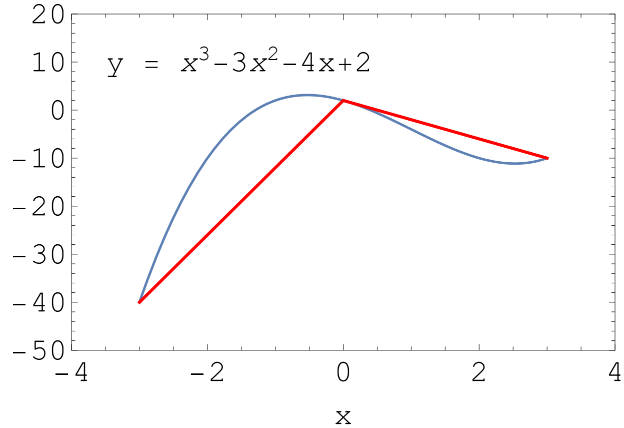
Figure 1: The curve we are interested in, $y = x^3 - 3x^2 - 4x + 2$, $-3 \leq x \leq 3$.

We can estimate the length of the curve by picking points on the curve, and joining the points with straight lines. This would be helpful since we know the straight line distance between two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

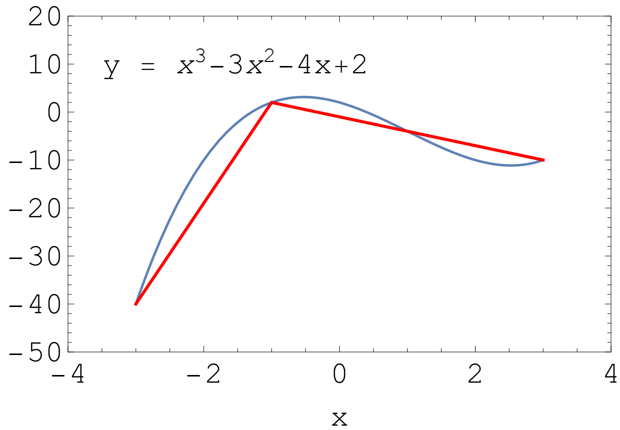
Although in principle the the distance between the x coordinates that we choose can differ between adjacent points, for simplicity let's choose a common distance for all the pairs of points, $x_i - x_{i-1} = \Delta x_i = \Delta x$. Figure 2 shows this procedure. We can see that once we get to 15 line segments, it becomes difficult to distinguish our straight lines from the curve itself!



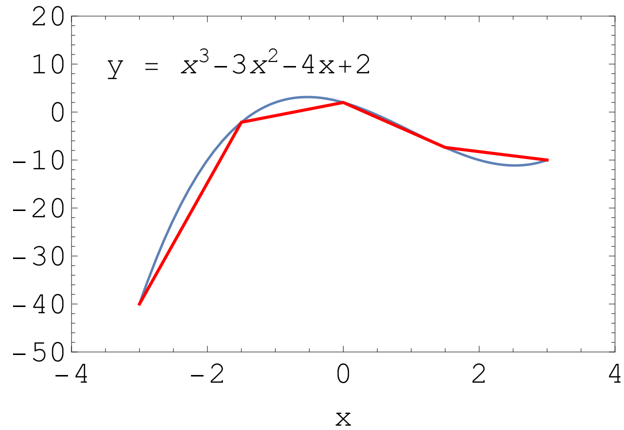
1 line segment, arc length ∼ 30.5941



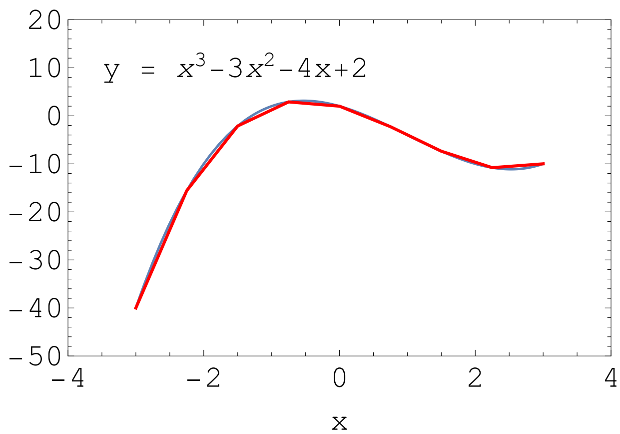
2 line segments, arc length ∼ 54.4763



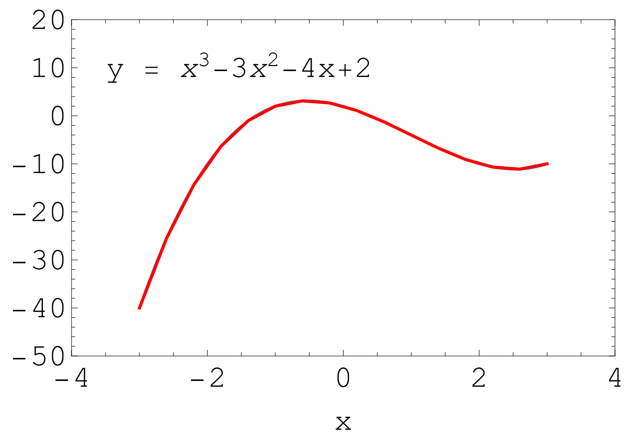
3 line segment, arc length ∼ 54.6967



4 line segments, arc length ∼ 54.8115



8 line segment, arc length ∼ 58.2357



15 line segments, arc length ∼ 59.1591

Figure 2: Using straight lines to approximate the length of the curve. The total length of the straight lines is also provided.

Integral Formulation

Here we show how we can write the length of the curve as an integral.

First, let's think about the length of one of the straight lines, a general line segment i with endpoints (x_{i-1}, y_{i-1}) and (x_i, y_i). From the formula for distance between two points, we have that its length is

$$\text{length of line segment } i = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

If we have n of these segments, we have to add them all up to get our approximation to the length of the entire curve,

$$\text{length of all line segments } = \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_i)^2} \sim \text{arc length of curve.} \tag{1}$$

Equation (1) was used to calculate the approximate arc lengths in Figure 2.

If we take the limit of the number of line segments to infinity, we get,

$$\text{length of all line segments } = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \text{arc length of curve.}$$

To write this as an integral using the Leibneiz notation we are familiar with, we have to do a bit of work. The problem is that we don't have an exposed factor at the end of the form Δx, which we need if we are to write this as an integral with respect to x. The manipulations we need to perform, however, are not difficult.

$$\begin{aligned} \text{arc length of curve} &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_i)^2} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \sqrt{(\Delta x)^2} \end{aligned}$$

Since $\Delta x > 0$, we can write $\sqrt{(\Delta x)^2} = \Delta x$, and now we are able to write this in the usual form for integration:

$$L = \text{arc length of curve } = \int_{-3}^{3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

This is generalized to the curve $y = f(x)$, $a \le x \le b$ as

$$L = \int_a^b ds,$$

where:

$$ds = \sqrt{1 + (f'(x))^2}\, dx,$$

or

$$(ds)^2 = (dx)^2 + (dy)^2.$$

For the example we have shown, the arc length integral is *very difficult* to calculate in a closed form!

An Incorrect Formulation of Arc Length

Let's examine an incorrect formulation of arc length, and see what we can learn. We'll use the same example as in the previous section.

Let's estimate the length of the curve by picking only the tops of the rectangles we used to write

$$\text{Area under curve } f(x),\ a \leq x \leq b = \int_a^b f(x)\, dx$$

via the right hand rule. This may seem like a good idea, but we shall see that it is incorrect.

Although in principle the distance between the x coordinates that we choose can differ between adjacent points, for simplicity let's choose a common distance for all the pairs of points as we did before. Figure 3 shows this procedure.

Our line segments are not approximating the function very well. We might think that as we took an infinite number of these lines, they would shrink in size until they became "points", and we would be left with only the points on the curve, and so we should get the length of the curve; this is not what happens. Let's look at the Riemann sum in this case and see what we are actually getting (you may have already guessed!).

Integral Formulation

Here we show what integral the procedure above leads to.

First, let's think about the length of one of the straight lines, a general line segment i with endpoints (x_{i-1}, y_{i-1}) and (x_i, y_i). For the horizontal straight lines we have, the length of the straight lines is always

$$\text{length of horizontal line segment } i = x_i - x_{i-1} = \Delta x.$$

If we have n of these segments, we have to add them all up

$$\text{length of all horizontal line segments } = \sum_{i=1}^{n} \Delta x \tag{2}$$

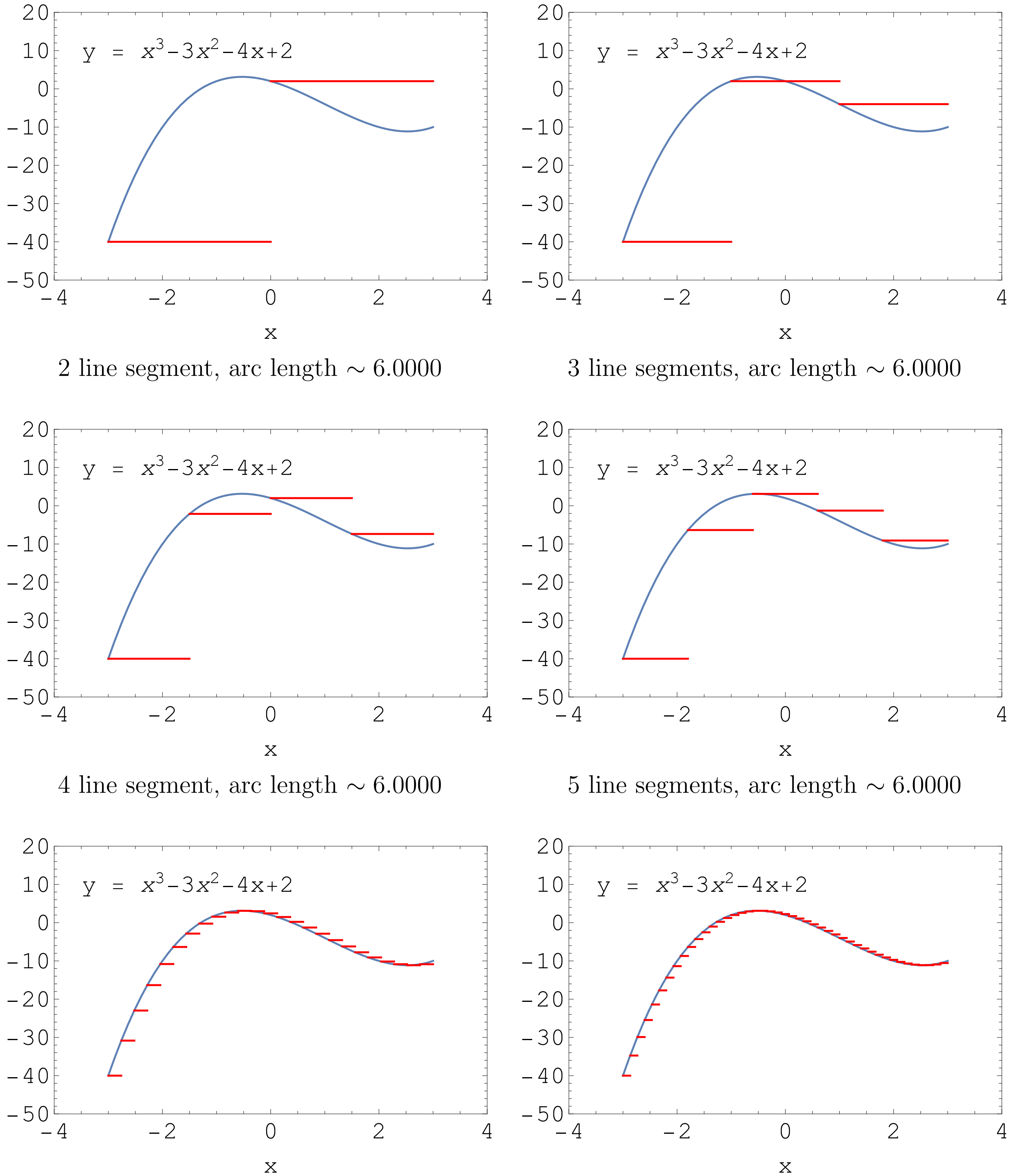
Equation (2) was used to calculate the lengths in Figure 3.

If we take the limit of the number of line segments to infinity, we get,

$$\text{length of all horizontal line segments } = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x = \int_{-3}^{3} dx.$$

This is in a form that converts to the Leibneiz notation automatically, so we've done that. We see that this integral is not equal to the arc length of the curve–it is simply equal to the length of the interval in x (6 in this case)!

By taking only horizontal lines, we have completely left out of the problem the length of the curve in the y direction, and hence we do not obtain an integral formula for arc length of a curve.



2 line segment, arc length ∼ 6.0000

3 line segments, arc length ∼ 6.0000

4 line segment, arc length ∼ 6.0000

5 line segments, arc length ∼ 6.0000

25 line segment, arc length ∼ 6.0000

45 line segments, arc length ∼ 6.0000

Figure 3: Using straight horizontal lines to approximate the length of the curve. The total length of the straight lines is also provided, and we see it does not represent the length of the curve.

Surface Area of Revolution

When a function is revolved about the x axis to create a surface, and we wish to find the surface area of that surface, we may think that we could find the surface area by evaluating the integral $\int_a^b 2\pi y\,dx$ (through cylinders), but we would find that we get the wrong answer.

We need to use the arc length to approximate along the curve, which leads to a shape which is the frustum of a cone rather than a cylinder. This is illustrated in Fig. 4.

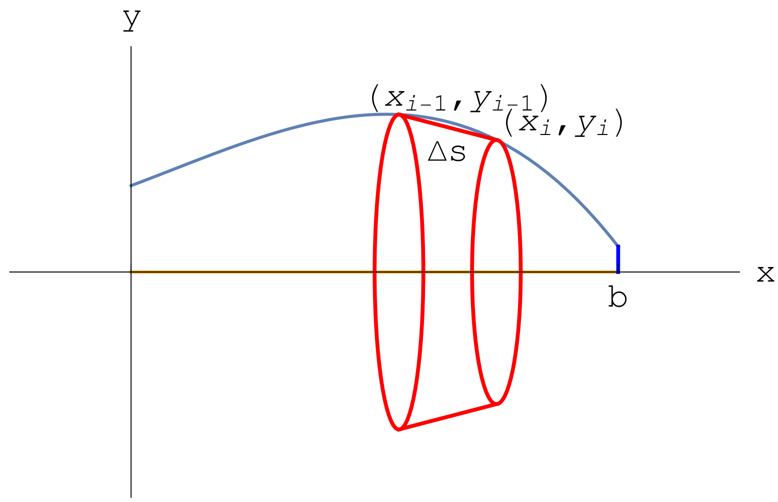


Figure 4: The frustum of a cone that is used for surface of revolution. Only one representative frustum is drawn.

Integral Formulation

Here we show how we can write the surface area as an integral. We need the surface area of the frustum, which is a function of the radii of the two circles and the distance Δs, and is given by:

$$\text{surface area of frustum } i = \pi(y_{i-1} + y_i)\,\Delta s.$$

If we have n of these frustums, we have to add them all up to get our approximation to the entire surface area,

$$\text{surface area of all frustums } = \sum_{i=1}^{n} \pi(y_{i-1} + y_i)\,\Delta s \sim \text{surface area of surface of revolution.}$$

If we take the limit of the number of frustums to infinity, we get,

$$\text{surface area of all frustums } = \lim_{n\to\infty} \sum_{i=1}^{n} \pi(y_{i-1} + y_i)\,\Delta s = \text{surface area of surface of revolution.}$$

$$\begin{aligned}\text{surface area of surface of revolution} &= \lim_{n\to\infty} \sum_{i=1}^{n} \pi(y_{i-1} + y_i)\,\Delta s \\ &= \int_a^b 2\pi f(x)\,ds\end{aligned}$$

The integration is over ds, where $(ds)^2 = (dx)^2 + (dy)^2$. We get the final formula since as $n \longrightarrow \infty$, $y_{i-1} + y_i \longrightarrow 2y_i$.