In **Partial Fractions** we assume a particular form that the split function of $f(x) = P(x)/Q(x)$ will take. When we assume a form for the solution, that assumption is justified only if it works! This is a very common technique in applied mathematics.

Case IV $Q(x)$ contains irreducible quadratic factors, some of which are repeated. I will not be testing you on these types.

Case I $Q(x)$ is a product of distinct linear factors.

Example $\dfrac{1}{(t+4)(t-1)}$

Here $Q(x) = (t+4)(t-1)$, a product of distinct linear factors. The degree of $Q(x)$ is 2, which is greater than the degree of $P(x)$, which is 0. We can use partial fractions directly, without dividing first.

$$
\begin{aligned}
\frac{1}{(t+4)(t-1)} &= \frac{1}{(t+4)(t-1)} && \text{Factor (already done)} \\
&= \frac{A}{(t+4)} + \frac{B}{(t-1)} && \text{Split} \\
1 &= A(t-1) + B(t+4) && \text{Clearing Fractions} \\
\\
1 &= A(-4-1) + B(-4+4) && \text{To determine } A\text{: evaluate at } t = -4 \\
1 &= A(-5) + B(0) \\
A &= -\frac{1}{5} \\
\\
1 &= A(+1-1) + B(+1+4) && \text{To determine } B\text{: evaluate at } t = +1 \\
1 &= A(0) + B(5) \\
B &= \frac{1}{5} \\
\frac{1}{(t+4)(t-1)} &= -\frac{1/5}{(t+4)} + \frac{1/5}{(t-1)}
\end{aligned}
$$

Case II $Q(x)$ is a product of linear factors, some of which are repeated.

Example $\dfrac{x^2}{(x+1)^3}$

Here $Q(x) = (x+1)^3$, a product of a linear factor which is repeated three times. The degree of $Q(x)$ is 3, which is greater than the degree of $P(x)$, which is 2. We can use partial fractions directly, without dividing first.

$$
\begin{aligned}
\frac{x^2}{(x+1)^3} &= \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} && \text{Split} \\
x^2 &= A(x+1)^2 + B(x+1) + C && \text{Clearing Fractions} \\
\\
(-1)^2 &= A(-1+1)^2 + B(-1+1) + C && \text{To determine } C\text{: evaluate at } x = -1 \\
1 &= A(0) + B(0) + C \\
C &= 1 \\
\\
x^2 &= A(x+1)^2 + B(x+1) + 1 \\
(x+1)(x-1) &= A(x+1)^2 + B(x+1) \\
x - 1 &= A(x+1) + B \\
\\
-1 - 1 &= A(-1+1) + B && \text{To determine } B\text{: evaluate at } x = -1 \\
-2 &= A(0) + B \\
B &= -2 \\
\\
x - 1 &= A(x+1) - 2 \\
0 - 1 &= A(0+1) - 2 && \text{To determine } A\text{: evaluate at } x = 0 \\
A &= +1 \\
\\
\frac{x^2}{(x+1)^3} &= \frac{1}{(x+1)} + \frac{(-2)}{(x+1)^2} + \frac{1}{(x+1)^3}
\end{aligned}
$$

NOTE: The trick to get the coefficients easily by evaluating at specific x does not always work in this case. You can always collect powers of x and solve the resulting equations instead.

Case III $Q(x)$ contains irreducible quadratic factors, none of which are repeated.

Example $\dfrac{x^2}{(x^3+1)}$

Here $Q(x) = (x^3+1) = (x+1)(1-x+x^2)$, a product of a linear factor and a quadratic factor which is not repeated. The degree of $Q(x)$ is 3, which is greater than the degree of $P(x)$, which is 2. We can use partial fractions directly, without dividing first. In this example, I will collect powers of x to get the equations to solve, which is just showing you another way of doing partial fractions.

$$\begin{aligned}
\frac{x^2}{(x^3+1)} &= \frac{x^2}{(x+1)(1-x+x^2)} && \text{Factor} \\
&= \frac{A}{(x+1)} + \frac{Bx+C}{(1-x+x^2)} && \text{Split} \\
x^2 &= A(1-x+x^2) + (Bx+C)(x+1) && \text{Clearing Fractions} \\
0 &= -x^2 + A - Ax + Ax^2 + Bx^2 + Cx + Bx + C && \text{Collect Powers of } x \\
0 &= (A+C)x^0 + (-A+C+B)x^1 + (-1+A+B)x^2
\end{aligned}$$

So we have to solve the system of equations for A, B, C:

$$A + C = 0 \tag{1}$$
$$-A + C + B = 0 \tag{2}$$
$$-1 + A + B = 0 \tag{3}$$

From Eq. (3), we have $A = 1 - B$. Then from Eq. (2) we have $C = -2B+1$, and Eq. (1) gives $-2B+1 = -1+B$, so $B = 2/3$, and $C = -1/3$, and $A = 1/3$.

$$\frac{x^2}{(x^3+1)} = \frac{1/3}{(x+1)} + \frac{(2x/3 - 1/3)}{(1-x+x^2)}$$

Completing the Square $\displaystyle\int \frac{x+2}{(1+x+x^2)}\,dx$

Solution It can be done if we complete the square:

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4}$$

So we can write $1 + x + x^2 = x^2 + x + 1 = (x + \frac{1}{2})^2 - \frac{1}{4} + 1 = (x + 1/2)^2 + 3/4$.

$$\begin{aligned}
\int \frac{x+2}{(1+x+x^2)}\,dx &= \int \frac{x+2}{\left(\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\right)}\,dx \qquad \text{Substitution: } u = x + 1/2 \\
&= \int \frac{u - 1/2 + 2}{(u^2 + \frac{3}{4})}\,du \\
&= \int \frac{u - 3/2}{(u^2 + \frac{3}{4})}\,du \\
&= \int \frac{u}{(u^2 + \frac{3}{4})}\,du + \frac{3}{2}\int \frac{1}{(u^2 + \frac{3}{4})}\,du \qquad \text{Substitution: } t = u^2 + 3/4 \\
&= \frac{1}{2}\int \frac{dt}{t} + \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}u\right) \\
&= \frac{1}{2}\ln|t| + \sqrt{3}\arctan\left(\frac{2}{\sqrt{3}}u\right) + c \\
&= \frac{1}{2}\ln|1 + x + x^2| + \sqrt{3}\arctan\left(\frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)\right) + c
\end{aligned}$$

We can combine the partial fraction cases, which just makes for more equations to solve.

Example $\displaystyle\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}\,dx$

$$\frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2 + 1}$$

$$x^2 - 2x - 1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

Evaluate at $x = 1$: $-2 = 2B \longrightarrow B = -1$

Evaluate at $x = 0$: $-1 = -A + B + D \longrightarrow A = D$

Evaluate at $x = -1$: $2 = -4A + 2B - 4C + 4D \longrightarrow 1 = -A - C + D$

Evaluate at $x = 2$: $-1 = 5A + 5B + 2C + D \longrightarrow 4 = 5A + 2C + D$

You can solve the system of three equations and three unknowns by any method you like (back substitution works fine).

You will find $A = 1, C = -1, D = 1$.

Therefore,

$$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{1-x}{x^2+1}$$
$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^2}\,dx + \int \frac{1-x}{x^2+1}\,dx$$

The first and second integrals can be done using a substitution, the third should be split up to involve a substitution and an arctangent form.

$$\begin{aligned}
\int \frac{1}{x-1}\,dx &= \int \frac{du}{u} \quad \text{where } u = x-1,\ du = dx \\
&= \ln|u| + c_1 \\
&= \ln|x-1| + c_1 \\
\int \frac{1}{(x-1)^2}\,dx &= \int \frac{du}{u^2} \quad \text{where } u = x-1,\ du = dx \\
&= -\frac{1}{u} + c_2 \\
&= -\frac{1}{x-1} + c_2 \\
\int \frac{1-x}{x^2+1}\,dx &= \int \frac{1}{x^2+1}\,dx - \int \frac{x}{x^2+1}\,dx \\
&= \arctan x - \frac{1}{2}\int \frac{du}{u}\,dx \quad \text{where } u = x^2+1,\ du = 2xdx \\
&= \arctan x - \frac{1}{2}\ln|u| + c_3 \\
&= \arctan x - \frac{1}{2}\ln|x^2+1| + c_3
\end{aligned}$$

Putting it all together we get

$$\begin{aligned}
\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx &= \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^2}\,dx + \int \frac{1-x}{x^2+1}\,dx \\
&= \ln|x-1| + \frac{1}{x-1} + \arctan x - \frac{1}{2}\ln|x^2+1| + c
\end{aligned}$$

where $c = c_1 - c_2 + c_3$.