Example 11.8.4 For the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$, find the radius of convergence and the interval of convergence.

Let's use the ratio test, with $a_n = \dfrac{(-1)^n x^n}{n+1}$. It will tell us the radius of convergence.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+2}\cdot\frac{n+1}{(-1)^n x^n}\right| \\
&= \lim_{n\to\infty}\left|x^{n+1-n}\frac{n+1}{n+2}\right| \\
&= |x|\lim_{n\to\infty}\frac{n+1}{n+2} \\
&= |x|\lim_{n\to\infty}\frac{1+1/n}{1+2/n} \\
&= |x|\frac{1+0}{1+0} = |x|
\end{aligned}$$

So $|x| < 1$ for absolute convergence. The center of this series is $x = a = 0$, and the radius of convergence is $R = 1$.

We must check the endpoints separately to get the interval of convergence.

$x = 1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

We can test this series using the alternating series test. Identify $b_n = 1/(n+1)$.

Since $b_{n+1} = \dfrac{1}{n+2} < \dfrac{1}{n+1} = b_n$, the first condition is satisfied.

Since $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n+1} = 0$ the second condition is satisfied.

Therefore, the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges by the alternating series test.

$x = -1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{n+1} = \sum_{n=0}^{\infty}\frac{1}{n+1} = \sum_{m=1}^{\infty}\frac{1}{m}$ which is a p-series, with $p = 1$ so it diverges.

The interval of convergence for the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ is $I = (-1, 1]$.

Example 11.8.38 Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{2n}$?

Both these series have a center of $x = a = 0$.

Since $\sum c_n x^n$ has a radius of convergence R, we know that the series converges for all $|x - a| < R$, or $|x| < R$ in this case.

The second series is $\sum c_n x^{2n} = \sum c_n (x^2)^n$. This is the same form as the first series, with x replaced by x^2. Therefore the new series will have a radius of convergence which satisfies $|x^2| < R$, or $|x| < \sqrt{R}$.

The radius of convergence of the new series is $\sqrt{R}$.

Example For the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x+2)^n}{n+1}$, find the radius of convergence and the interval of convergence.

Let's use the ratio test, with $a_n = \dfrac{(-1)^n (x+2)^n}{n+1}$. It will tell us the radius of convergence.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x+2)^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n (x+2)^n} \right| \\
&= |x+2| \lim_{n\to\infty} \frac{n+1}{n+2} \\
&= |x+2| \lim_{n\to\infty} \frac{1+1/n}{1+2/n} \\
&= |x+2| \left(\frac{1+0}{1+0} \right) = |x+2|
\end{aligned}$$

Therefore, for the series to be absolutely convergent, we require $|x+2| < 1$. From this, we can say the series has a center of $a = -2$, and a radius of convergence of $R = 1$.

We need to check the endpoints separately. The endpoints of the region are found by expanding $|x+2| < 1$:

$$\begin{aligned}
-1 < \; & x+2 & < 1 \\
-1-2 < \; & x & < 1-2 \\
-3 < \; & x & < -1
\end{aligned}$$

$x = -3$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (-3+2)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$. This is the divergent harmonic series (p-series with $p = 1$).

$x = -1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (-1+2)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$.

We can test this series using the alternating series test. Identify $b_n = 1/(n+1)$.

Since $b_{n+1} = \dfrac{1}{n+2} < \dfrac{1}{n+1} = b_n$, the first condition is satisfied.

Since $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$ the second condition is satisfied.

Therefore, the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges by the alternating series test.

The interval of convergence for the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x+2)^n}{n+1}$ is $I = (-3, -1]$.