1102 Calculus II 11.7 Strategy for Testing Series

Example 11.7.1 Is the series $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \frac{n^2-1}{n^2+n}$.

Since the power of n is the same in the numerator and denominator, we expect the limit as $n \to \infty$ of a_n not to be zero. Therefore, let's use the test for divergence.

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n^2-1}{n^2+n} \\ &= \lim_{n\to\infty} \frac{1-1/n^2}{1+1/n} \\ &= \frac{1-0}{1+0} = 1 \neq 0 \end{aligned}$$

Since $\lim_{n\to\infty} a_n \neq 0$, the series $\sum a_n$ diverges by the test for divergence.

Example 11.7.4 Is the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = (-1)^{n-1} \frac{n-1}{n^2+n}$.

Since the series is alternating, we should try the alternating series test.

We will also need $b_n = |a_n| = \frac{n-1}{n^2+n}$.

First, we need to show $b_{n+1} \leq b_n$ for all n. We could try to do it by inspection, like we have done before,

$$b_{n+1} = \frac{n}{(n+1)^2+n+1}$$

but it is hard to compare this to b_n. Instead, we can look at the continuous counterpart. If the derivative is less than zero, then the function is decreasing, and we will be able to say that $b_{n+1} \leq b_n$.

$$\begin{aligned} f(x) &= \frac{x-1}{x^2+x} \\ f'(x) &= \frac{(x^2+x)(1)-(x-1)(2x+1)}{(x^2+x)^2} \\ &= \frac{(x^2+x)(1)-(x-1)(2x+1)}{(x^2+x)^2} \\ &= \frac{x^2+x-2x^2+2x-x+1}{(x^2+x)^2} \\ &= \frac{1+2x-x^2}{(x^2+x)^2} \end{aligned}$$

Now, the denominator is always greater than zero. For the numerator, we need to know when it becomes negative.

$$\begin{aligned}(1+2x-x^2)_{x=1} &= 1+2-1=2>0\\ (1+2x-x^2)_{x=2} &= 1+4-4=1>0\\ (1+2x-x^2)_{x=3} &= 1+6-9=-2<0\end{aligned}$$

So we have $f'(x)<0$ for $n\geq 3$. This means $b_{n+1}\leq b_n$ for $n\geq 3$.

The second condition of the alternating series test is that $\lim_{n\to\infty} b_n = 0$.

$$\begin{aligned}\lim_{n\to\infty} b_n &= \lim_{n\to\infty}\frac{n-1}{n^2+n}\\ &= \lim_{n\to\infty}\frac{1/n-1/n^2}{1+1/n}\\ &= \frac{0-0}{1+0}=0\end{aligned}$$

So the series $\displaystyle\sum_{n=3}^{\infty} a_n$ converges by the alternating series test.

Since $\displaystyle\sum_{n=3}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Now, we want to check the convergence of the series $\sum b_n = \sum |a_n|$, to determine if the series $\sum a_n$ is absolutely or conditionally convergent.

Let's check this series using the limit comparison test. Since $n-1\sim n$ and $n^2+n\sim n^2$ for large n, we should use as our comparison series $\sum c_n$ where $c_n = n/n^2 = 1/n$, which is the divergent p-series with $p=1$.

$$\begin{aligned}\lim_{n\to\infty}\frac{c_n}{b_n} &= \lim_{n\to\infty}\frac{1}{n}\cdot\frac{n^2+n}{n-1}\\ &= \lim_{n\to\infty}\frac{n+1}{n-1}\\ &= \lim_{n\to\infty}\frac{1+1/n}{1-1/n}\\ &= \frac{1+0}{1-0}=1>0 \text{ and finite.}\end{aligned}$$

So, since the limit was greater than zero and finite, and the comparison series $\sum c_n$ diverged, the series $\sum b_n = \sum |a_n|$ must also diverge.

Therefore, $\sum a_n$ converges and $\sum |a_n|$ diverges, so the series $\sum a_n$ is conditionally convergent.

Example 11.7.6 Is the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n}{1+8n}\right)^n$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \left(\dfrac{3n}{1+8n}\right)^n$.

Since we have $a_n = (b_n)^n$, we should try the root test.

$$\begin{aligned}
\lim_{n\to\infty}(|a_n|)^{1/n} &= \lim_{n\to\infty}\left(\left|\left(\frac{3n}{1+8n}\right)^n\right|\right)^{1/n} \\
&= \lim_{n\to\infty}\left(\left(\frac{3n}{1+8n}\right)^n\right)^{1/n} \\
&= \lim_{n\to\infty}\frac{3n}{1+8n} \\
&= \lim_{n\to\infty}\frac{3}{1/n+8} \\
&= \frac{3}{0+8} = \frac{3}{8} < 1
\end{aligned}$$

The series $\sum a_n$ is absolutely convergent by the root test.

Example 11.7.9 Is the series $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^n}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \dfrac{n}{e^n}$.

Since we have a constant raised to the power n, we should try the ratio test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{n+1}{e^{n+1}}\cdot\frac{e^n}{n} \\
&= \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{1}{e^{n+1-n}} \\
&= \frac{1}{e}\lim_{n\to\infty}\frac{n+1}{n} \\
&= \frac{1}{e}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) \\
&= \frac{1}{e} < 1
\end{aligned}$$

The series $\sum a_n$ is absolutely convergent by the ratio test.

Example 11.7.14 Is the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+1}{n^3+1}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \dfrac{n^2+1}{n^3+1}$.

Because we have a ratio of polynomials, we should use one of the comparison tests. Since $n^2+1 \sim n^2$ and $n^3+1 \sim n^3$ for large n, we should use as our comparison series

$$\frac{n^2+1}{n^3+1} \sim \frac{n^2}{n^3} = \frac{1}{n}$$

Let $b_n = 1/n$, the divergent p-series with $p=1$.

Use the limit comparison test.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2+1}{n^3+1} \cdot n \\
&= \lim_{n\to\infty} \frac{n^3+n}{n^3+1} \\
&= \lim_{n\to\infty} \frac{1+1/n^2}{1+1/n^3} \\
&= \frac{1+0}{1+0} = 1 > 0 \text{ and finite.}
\end{aligned}$$

So, since the limit was greater than zero and finite, and the comparison series $\sum b_n$ diverged, the series $\sum a_n$ must also diverge.

Example 11.7.17 Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n+n}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \dfrac{3^n}{5^n+n}$.

Since we have a constant to a power of n appearing, we may want to try the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3^{n+1}}{5^{n+1}+n+1} \cdot \frac{5^n+n}{3^n}$$

This is yucky looking. It is yucky looking since the term 5^n+n will not easily simplify with its $n+1$ counterpart.

Let's try something different instead. A comparison test might work well.

$$a_n = \frac{3^n}{5^n+n} \leq \frac{3^n}{5^n} = \left(\frac{3}{5}\right)^n = b_n \quad \text{for all } n.$$

Our comparison series is $\sum b_n$, which is a geometric series with $a=1$ and $r=3/5$, and since $|r| = |3/5| < 1$ the series $\sum b_n$ converges.

Since $a_n \leq b_n$ for all n, and $\sum b_n$ converges, the series $\sum a_n$ converges by the comparison test.

Since the terms a_n are all positive, $a_n = |a_n|$. Therefore the series $\sum |a_n|$ is also convergent. Therefore, the series $\sum a_n$ is absolutely convergent.

Example 11.7.23 Is the series $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = (-1)^n 2^{1/n}$.

This series diverges by the test for divergence.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n 2^{1/n}$$

The $2^{1/n} \to 2^0 = 1$ as $n \to \infty$, but the alternating part $(-1)^n$ means that for large n the series $\sum a_n$ oscillates between (almost) ± 1. Therefore, $\lim_{n\to\infty} a_n$ does not exist, and the series $\sum a_n$ diverges by the test for divergence.

Example 11.7.25 Is the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = (-1)^n \frac{\ln n}{\sqrt{n}}$.

Since this is an alternating series test, we should try the alternating series test.

We identify $b_n = |a_n| = \frac{\ln n}{\sqrt{n}}$.

For the first condition of the alternating series test, $b_{n+1} \le b_n$ for all n, we need to work with the continuous function $f(x)$ where $f(n) = b_n$ and then show $f(x)$ is decreasing.

$$\begin{aligned} f(x) &= \frac{\ln x}{\sqrt{x}} \\ f'(x) &= \frac{\sqrt{x}\frac{1}{x} - \ln x \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} \\ &= \frac{2 - \ln x}{2x^{3/2}} \end{aligned}$$

So we have $f(x) < 0$ if $2 - \ln x < 0$, which is the same as $x > e^2$. This means that for $n > e^2$, $b_{n+1} \le b_n$.

I know $e < 3$, so let's just say that we have $b_{n+1} \le b_n$ if $n \ge 9$.

The second condition of the alternating series test is to show $\lim_{n\to\infty} b_n = 0$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \longrightarrow \frac{\infty}{\infty} \quad \text{indeterminate quotient}$$

We want to use L'Hospital's Rule to evaluate this limit, but we have to switch to a continuous function first, since the derivative is not defined for the discrete variable n. Pick $f(x)$ such that $f(n) = b_n$, and then proceed.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \longrightarrow \frac{\infty}{\infty} \quad \text{indeterminate quotient}$$

$$\begin{aligned} &= \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{2\sqrt{x}}\right)} \\ &= \lim_{x\to\infty} \frac{2}{\sqrt{x}} \\ &= 0 \end{aligned}$$

Therefore, we also have $\lim\limits_{n\to\infty} b_n = 0$.

The series $\sum\limits_{n=9}^{\infty} a_n$ converges by the alternating series test.

Since $\sum\limits_{n=9}^{\infty} a_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

Now, we want to check the convergence of the series $\sum b_n = \sum |a_n|$, to determine if the series $\sum a_n$ is absolutely or conditionally convergent.

We can use the comparison test for this. Here is how we construct our comparison series.

$$\begin{aligned} \ln n > 1 &\quad \text{if} \quad n > 3 \\ \frac{\ln n}{\sqrt{n}} > \frac{1}{\sqrt{n}} &\quad \text{if} \quad n > 3 \end{aligned}$$

So our comparison series should be $\sum c_n$ where $c_n = \dfrac{1}{\sqrt{n}}$, which is the divergent p-series, with $p = 1/2$.

Since $b_n = \dfrac{\ln n}{\sqrt{n}} > \dfrac{1}{\sqrt{n}} = c_n$ if $n > 3$, and $\sum c_n$ diverges, we have that $\sum b_n$ must also diverge by the comparison test.

Therefore, $\sum a_n$ converges and $\sum |a_n| = \sum b_n$ diverges, so the series $\sum a_n$ is conditionally convergent.

Example 11.7.31 Is the series $\sum\limits_{n=1}^{\infty} \dfrac{2^n}{(2n+1)!}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \dfrac{2^n}{(2n+1)!}$.

Since there is a factorial in a_n, we will try the ratio test.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{2^{n+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{2^n} \\ &= \lim_{n\to\infty} 2^{n+1-n} \frac{(2n+1)!}{(2n+3)!} \\ &= 2 \lim_{n\to\infty} \frac{1 \cdot 2 \cdot 3 \cdots (2n-1) \cdot (2n) \cdot (2n+1)}{1 \cdot 2 \cdot 3 \cdots (2n-1) \cdot (2n) \cdot (2n+1) \cdot (2n+2) \cdot (2n+3)} \\ &= 2 \lim_{n\to\infty} \frac{1}{(2n+2) \cdot (2n+3)} \end{aligned}$$

$$= \quad 0 < 1$$

The series $\sum a_n$ is absolutely convergent by the ratio test.

Example 11.7.30 Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \dfrac{e^{1/n}}{n^2}$.

Looking at the form of a_n, we notice that if $u = 1/x$, then $du = -1/x^2\,dx$, so we could probably use the integral test.

Let $f(x) = \dfrac{e^{1/x}}{x^2}$, and then we have $f(n) = a_n$.

On the interval $[1, \infty)$, we have that $f(x)$ is

- continuous, since $e^{1/x}$ and $1/x^2$ are both continuous,
- positive, since $e^{1/x}$ and $1/x^2$ are both positive,
- decreasing, since $e^{1/x}$ and $1/x^2$ are both decreasing.

So the integral test can be applied.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{e^{1/x}}{x^2}\,dx \quad \text{substitution:} \quad \begin{array}{l} u = 1/x \\ du = -1/x^2\,dx \\ \text{when } x = 1 \to u = 1 \\ \text{when } x = \infty \to u = 0 \end{array}$$

$$= -\int_1^{0} e^u\,du$$

$$= \int_0^{1} e^u\,du$$

$$= e^u\big|_0^1 = e^1 - e^0 = e - 1$$

Since the integral converges, the series $\sum a_n$ also converges by the integral test.

Since the terms a_n are greater than zero, we have $a_n = |a_n|$, and the series $\sum |a_n|$ converges as well. Therefore, the series $\sum a_n$ is absolutely convergent.