Questions

Example Determine $\displaystyle\int x \cos x \, dx$.

Example $\int e^{\theta} \cos \theta \, d\theta$

Example You may wonder why we do not add a constant at the point where we integrate for v in the parts substitution. This is because any constant added there will cancel later. Redo $\int x \cos x \, dx$ and keep all constants of integration to show this.

Example: $\displaystyle\int_0^{1/2} \arccos x \, dx$.

Example Prove the following integral formula (# 101 in table) using parts:

$$\int w^n \ln w \, dw = \frac{w^{n+1}}{(n+1)^2} \left[(n+1) \ln w - 1\right] + c$$

Solutions

Example Determine $\displaystyle\int x \cos x \, dx$.

Let's do this integral using parts. The formula for parts is $\displaystyle\int u dv = uv - \int v du$, so we want to choose the two substitutions for u and dv based on our integral. Let's choose

$$\begin{array}{ll} u = x & dv = \cos x \, dx \\ du = dx & v = \displaystyle\int \cos x \, dx = \sin x \end{array}$$

NOTE: once you have chosen u, dv has to be everything else that's left so we can write:

$$\begin{aligned} \int x \cos x \, dx &= \int u \, dv \qquad \text{(substitution)} \\ &= uv - \int v \, du \qquad \text{(parts)} \\ &= x \sin x - \int \sin x \, dx \qquad \text{(back substitution)} \\ &= x \sin x - (-\cos x) + c \qquad \text{(integrate)} \\ &= x \sin x + \cos x + c \qquad \text{(simplify)} \end{aligned}$$

What we have done is replaced the original integral, which we could not do, with the integral $\displaystyle\int \sin x \, dx$, which we could do. This is the power of parts!

Check our answer by taking the derivative, and verifying that we get the integrand back:

$$\frac{d}{dx}[x \sin x + \cos x + c] = \sin x + x \cos x - \sin x = x \cos x$$

The problem most people have with parts is picking the u and dv correctly. Keep in mind that we want to be able to do our new integral, so generally we pick u to be a function that gets simpler when differentiated. What would happen if we had chosen differently in the above? Let's choose

$$\begin{array}{ll} u = \cos x & dv = x\,dx \\ du = -\sin x\,dx & v = \displaystyle\int x\,dx = \frac{x^2}{2} \end{array}$$

$$\begin{aligned} \int x \cos x\,dx &= \int u\,dv \qquad \text{(substitution)} \\ &= uv - \int v\,du \qquad \text{(parts)} \\ &= \frac{x^2}{2}\cos x + \int \frac{x^2}{2}\sin x\,dx \qquad \text{(back substitution)} \end{aligned}$$

This is true, but it doesn't help us do the integral!

Sometimes you have to use parts twice, and sometimes you never have to integrate explicitly to get the integral.

Example $\int e^\theta \cos\theta\,d\theta$

We want to use parts, but neither e^θ nor $\cos\theta$ will simplify when we take a derivative. We have to do something, so let's try

$$\begin{array}{ll} u = \cos\theta & dv = e^\theta\,d\theta \\ du = -\sin\theta\,d\theta & v = \int e^\theta\,d\theta = e^\theta \end{array}$$

So the integral becomes:

$$\begin{aligned} \int e^\theta \cos\theta\,d\theta &= \int u\,dv \qquad \text{(substitution)} \\ &= uv - \int v\,du \qquad \text{(parts)} \\ &= e^\theta \cos\theta - \int e^\theta(-\sin\theta)\,d\theta \qquad \text{(back substitution)} \\ &= e^\theta \cos\theta + \int e^\theta \sin\theta\,d\theta \qquad \text{(simplify)} \qquad (1) \end{aligned}$$

Our new integral is no simpler. But, it is no worse! That tells us to try parts again, on the the integral:

$$\int e^\theta \sin\theta\,d\theta$$

Let's choose

$$\begin{array}{ll} u = \sin\theta & dv = e^\theta\, d\theta \\ du = \cos\theta\, d\theta & v = \int e^\theta\, d\theta = e^\theta \end{array}$$

NOTE: if we switch our choices of u and dv here, we will get the original integral back. So the integral becomes:

$$\begin{aligned} \int e^\theta \sin\theta\, d\theta &= \int u\, dv \quad \text{(substitution)} \\ &= uv - \int v\, du \quad \text{(parts)} \\ &= e^\theta \sin\theta - \int e^\theta \cos\theta\, d\theta \quad \text{(back substitution)} \end{aligned}$$

Now, we can substitute everything back into Equation (1):

$$\int e^\theta \cos\theta d\theta = e^\theta \cos\theta + \int e^\theta \sin\theta\, d\theta = e^\theta \cos\theta + e^\theta \sin\theta - \int e^\theta \cos\theta\, d\theta$$

Solve for the integral algebraically:

$$\int e^\theta \cos\theta\, d\theta = \frac{1}{2}[e^\theta \cos\theta + e^\theta \sin\theta] + c$$

Notice how we simply added a constant of integration at the end, since we had a definite integral.

Example You may wonder why we do not add a constant at the point where we integrate for v in the parts substitution. This is because any constant added there will cancel later. Let's redo $\int x\cos x\, dx$ to show this. Let's choose

$$\begin{array}{ll} u = x & dv = \cos x\, dx \\ du = dx & v = \int \cos x\, dx = \sin x + c_1 \end{array}$$

$$\begin{aligned} \int x\cos x\, dx &= \int u\, dv \quad \text{(substitution)} \\ &= uv - \int v\, du \quad \text{(parts)} \\ &= x(\sin x + c_1) - \int (\sin x + c_1)\, dx \quad \text{(back substitution)} \\ &= x\sin x + xc_1 - [(-\cos x) + c_1 x] + c \quad \text{(integrate)} \\ &= x\sin x + \cos x + c \quad \text{(simplify)} \end{aligned}$$

Parts is also used to prove many of the integral formulas in the back of your textbook, and especially the ones that are written in terms of reduction formulas. The following example shows where #88 from the table of integrals comes from.

Definite Integrals by Parts: $\displaystyle\int_a^b u\, dv = uv|_a^b - \int_a^b v\, du.$

Example: $\int_0^{1/2} \arccos x\, dx$.

To begin, we use parts and choose:

$$\begin{array}{ll} u = \arccos x & dv = dx \\ du = \dfrac{-1}{\sqrt{1-x^2}}\, dx & v = x \end{array}$$

$$\begin{aligned}
\int_0^{1/2} \arccos x\, dx &= \int_0^{1/2} u\, dv \qquad \text{(substitution)} \\
&= uv|_0^{1/2} - \int_0^{1/2} v\, du \qquad \text{(parts)} \\
&= x \arccos x|_0^{1/2} - \int x \left(\frac{-1}{\sqrt{1-x^2}} \right) dx \qquad \text{(back substitution)} \\
&= \frac{1}{2} \arccos \frac{1}{2} - 0 \arccos 0 + \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\, dx \\
&\qquad \text{Substitution: } \begin{array}{ll} t = 1 - x^2 & \text{when } x = 0 \to t = 1 \\ dt = -2x dx & \text{when } x = \frac{1}{2} \to t = \frac{3}{4} \end{array} \\
&= \frac{1}{2}\left(\frac{\pi}{3}\right) - 0\left(\frac{\pi}{2}\right) + \int_1^{3/4} \frac{1}{\sqrt{t}} \frac{dt}{(-2)} \\
&= \frac{\pi}{6} - \frac{1}{2} \int_1^{3/4} t^{-1/2}\, dt \\
&= \frac{\pi}{6} - \frac{1}{2} \left. \frac{t^{1/2}}{(1/2)} \right|_1^{3/4} \\
&= \frac{\pi}{6} - (\sqrt{3/4} - \sqrt{1}) \\
&= \frac{\pi}{6} + 1 - \sqrt{3/4}
\end{aligned}$$

Example Prove the following integral formula (# 101 in table) using parts:

$$\int w^n \ln w\, dw = \frac{w^{n+1}}{(n+1)^2} [(n+1) \ln w - 1] + c$$

To begin, we use parts and choose:

$$\begin{array}{ll} u = \ln w & dv = w^n\, dw \\ du = \dfrac{1}{w}\, dw & v = \displaystyle\int w^n\, dw = \frac{w^{n+1}}{(n+1)} \end{array}$$

Notice that the above integral for v is only true if $n \neq -1$.

$$\begin{aligned}
\int w^n \ln w\, dw = &= \int u\, dv \qquad \text{(substitution)} \\
&= uv - \int v\, du \qquad \text{(parts)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{w^{n+1}}{(n+1)} \ln w - \int \frac{w^{n+1}}{(n+1)} \frac{1}{w}\, dw \qquad \text{(back substitution)} \\
&= \frac{w^{n+1}}{(n+1)} \ln w - \frac{1}{n+1} \int w^n \, dw \qquad \text{(simplification)} \\
&= \frac{w^{n+1}}{(n+1)} \ln w - \frac{1}{n+1} \left(\frac{w^{n+1}}{n+1} \right) + c \\
&= \frac{w^{n+1}}{(n+1)} \ln w - \frac{w^{n+1}}{(n+1)^2} + c \\
&= \frac{w^{n+1}}{(n+1)^2} \left[(n+1) \ln w - 1 \right] + c
\end{aligned}$$

We have shown $\displaystyle\int w^n \ln w \, dw = \frac{w^{n+1}}{(n+1)^2} \left[(n+1) \ln w - 1 \right] + c, \quad n \neq -1.$

If $n = -1$, it is not too difficult to show $\displaystyle\int \frac{\ln w}{w}\, dw = \frac{\ln^2 w}{2} + c.$