**Questions**

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**Example** Find the volume of the solid that is obtained by rotating the region bounded by $y = x^2$, $x = y^2$ about the line $x = -1$.

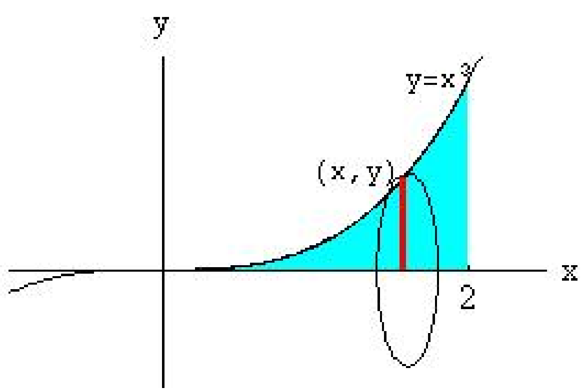
**Example** Find the volume of the solid that is obtained by rotating the region bounded by $y = x^2$, $x = y^2$ about the $x$-axis.

**Example** Find the volume of the solid that is the cap of a sphere with radius $r$ and height $h$.

**Solutions**

**Example** Find the volume of the solid that is obtained by rotating the region below $y = x^3$ and above $0 < x < 2$, about the $x$-axis.

**Solution** First, sketch the situation:

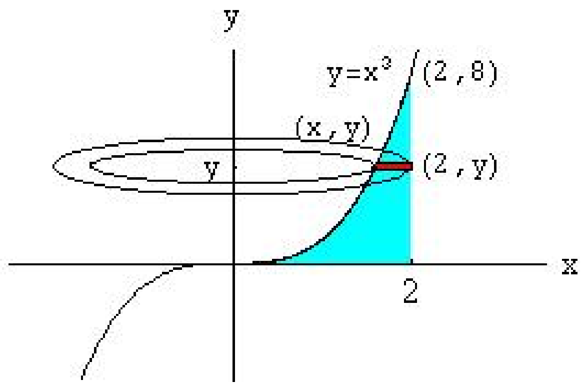



Integration limits: if $0 < x < 2$, the circle sweeps out the volume. We integrate with respect to $x$.
Radius of circle $= y = x^3$.
Area of circle $= \pi y^2 = \pi x^6$.

$$\text{Volume} = \int_a^b (\text{area of circle})\,dx = \pi \int_0^2 x^6\,dx = \pi \frac{x^7}{7}\Big|_0^2 = \frac{128\pi}{7}$$

**Example** Find the volume of the solid that is obtained by rotating the region below $y = x^3$ and above $0 < x < 2$, about the $y$-axis.

**Solution** First, sketch the situation:

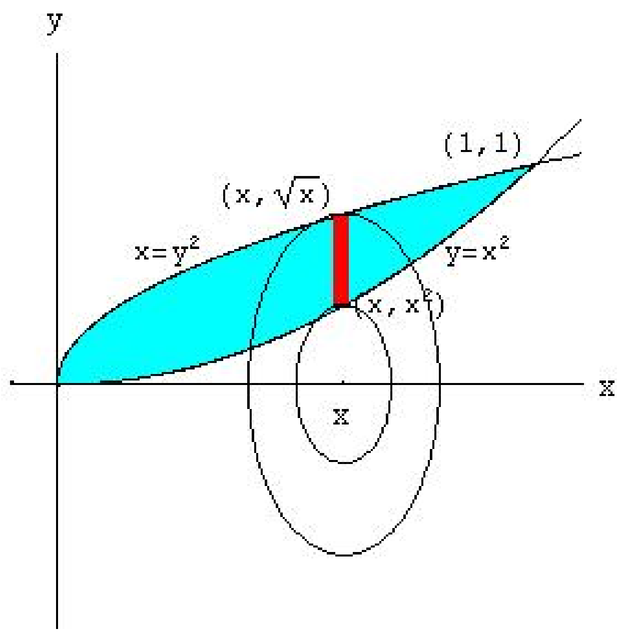



Integration limits: if $0 < y < 8$, the washer sweeps out the volume. We integrate with respect to $y$.
Radius of inner circle $= x = y^{1/3}$.
Radius of outer circle $= 2$.
Area of washer $= \pi[(\text{radius outer circle})^2 - (\text{radius of inner circle})^2] = \pi[4 - y^{2/3}]$.

$$\begin{aligned}\text{Volume} &= \int_c^d (\text{area of washer})\, dy = \pi \int_0^8 (4 - y^{2/3})\, dy = \pi \left(4y - \frac{y^{5/3}}{5/3}\right)_0^8 \\ &= \pi\left(32 - \frac{3}{5}8^{5/3}\right) = \pi\left(32 - \frac{96}{5}\right) = \frac{64\pi}{5}\end{aligned}$$

**Example** Find the volume of the solid that is obtained by rotating the region bounded by $y = x^2$, $x = y^2$ about the $x$-axis.

**Solution** First, sketch the situation:

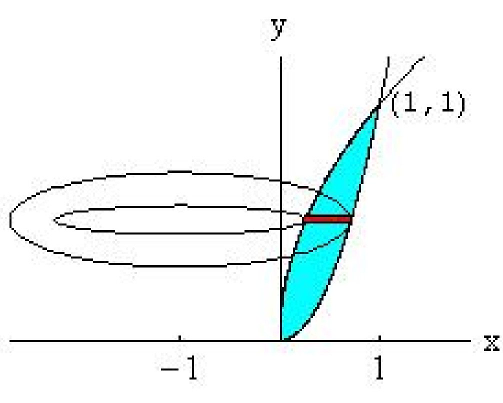



Integration limits: if $0 < x < 1$, the washer sweeps out the volume. We integrate with respect to $x$.
Radius of outer circle $= y = \sqrt{x}$.
Radius of inner circle $= y = x^2$.
Area of washer $= \pi[$ (outer radius)$^2$ - (inner radius)$^2$ $] = \pi(x - x^4)$.

$$\text{Volume} = \int_a^b (\text{area of washer})\,dx = \int_0^1 \pi(x - x^4)\,dx = \pi\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = \frac{3}{10}\pi$$

**Example** Find the volume of the solid that is obtained by rotating the region bounded by $y = x^2$, $x = y^2$ about the line $x = -1$.

**Solution** First, sketch the situation:




Integration limits: if $0 < y < 1$, the washer sweeps out the volume. We integrate with respect to $y$.
Radius of outer circle $= 1 + x = 1 + \sqrt{y}$.

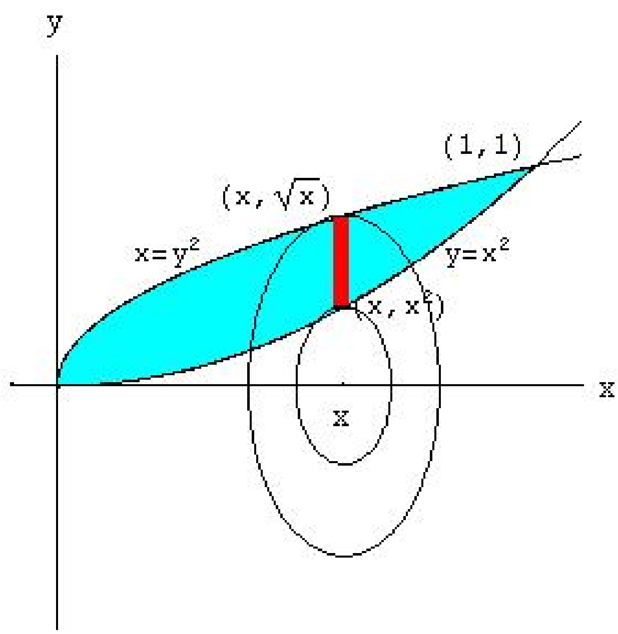
Radius of inner circle $= 1 + x = 1 + y^2$.
This creates a washer: area of washer $= \pi[$ (outer radius)$^2$ - (inner radius)$^2$ $] = \pi((1+\sqrt{y})^2 - (1+y^2)^2)$.

$$\text{Volume} = \int_0^1 \pi((1+\sqrt{y})^2 - (1+y^2)^2)\, dy$$

$$= \pi \int_0^1 \left(2\sqrt{y} + y - 2y^2 - y^4\right)\, dy = \pi \left(2\frac{y^{3/2}}{3/2} + \frac{y^2}{2} - 2\frac{y^3}{3} - \frac{y^5}{5}\right)_0^1 = \pi\left(\frac{4}{3} + \frac{1}{2} - \frac{2}{3} - \frac{1}{5}\right) = \frac{29}{30}\pi$$

**Example** Find the volume of the solid that is obtained by rotating the region bounded by $y = x^2$, $x = y^2$ about the $x$-axis.

**Solution** First, sketch the situation:

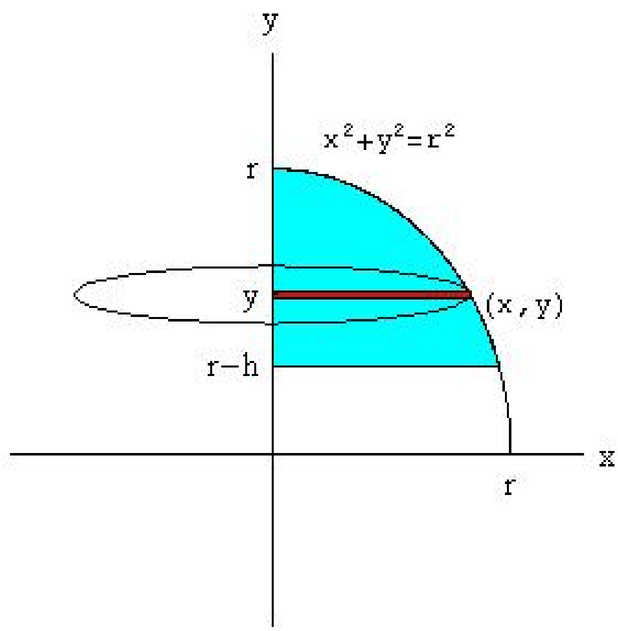



Integration limits: if $0 < x < 1$, the washer sweeps out the volume. We integrate with respect to $x$.
Radius of outer circle $= y = \sqrt{x}$.
Radius of inner circle $= y = x^2$.
Area of washer $= \pi[$ (outer radius)$^2$ - (inner radius)$^2$ $] = \pi(x - x^4)$.

$$\text{Volume} = \int_a^b (\text{area of washer})\, dx = \int_0^1 \pi(x - x^4)\, dx = \pi\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = \frac{3}{10}\pi$$

**Example** Find the volume of the solid that is the cap of a sphere with radius $r$ and height $h$.

**Solution** First, sketch the situation. Rotating the shaded region about the $y$ axis will produce the cap of the sphere with the required dimensions.




Integration limits: if $r - h < y < r$, the circle sweeps out the volume. We integrate with respect to $y$.
Radius of circle $= x = \sqrt{r^2 - y^2}$.
Area of circle $= \pi(\text{radius})^2 = \pi(r^2 - y^2)$.

$$\begin{aligned}
\text{Volume} &= \int_a^b (\text{area of circle})\, dy \\
&= \int_{r-h}^{r} \pi(r^2 - y^2)\, dy \\
&= \pi\left(r^2 y - \frac{y^3}{3}\right)_{r-h}^{r} \\
&= \pi\left[\left(r^2(r) - \frac{r^3}{3}\right) - \left(r^2(r-h) - \frac{(r-h)^3}{3}\right)\right] \\
&= \pi\left[\frac{2r^3}{3} - r^3 + r^2 h + \frac{1}{3}(r^3 - 3hr^2 + 3h^2 r - h^3)\right] \\
&= \pi\left[-\frac{r^3}{3} + r^2 h + \frac{r^3}{3} - hr^2 + h^2 r - \frac{h^3}{3}\right] \\
&= \pi\left[h^2 r - \frac{h^3}{3}\right] \\
&= \pi h^2\left[r - \frac{h}{3}\right]
\end{aligned}$$