Questions

Example Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$ absolutely convergent, conditionally convergent, or divergent?

Example Is the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$ absolutely convergent, conditionally convergent, or divergent?

Example Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{5+n}$ absolutely convergent, conditionally convergent, or divergent?

Example Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ absolutely convergent, conditionally convergent, or divergent?

Example Is the series $\displaystyle\sum_{n=1}^{\infty} e^{-n} n!$ absolutely convergent, conditionally convergent, or divergent?

Example Is the series $\displaystyle\sum_{n=2}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$ absolutely convergent, conditionally convergent, or divergent?

Example Is the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ absolutely convergent, conditionally convergent, or divergent?

Example Is the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ absolutely convergent, conditionally convergent, or divergent?

Example Show that $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x. Deduce that $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x.

Solutions

Example Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \dfrac{(-3)^n}{n^3}$.

The a_n contains a power involving n, so we should try the root test.

$$\begin{aligned}
\lim_{n\to\infty} (|a_n|)^{1/n} &= \lim_{n\to\infty} \left(\left|\frac{(-3)^n}{n^3}\right|\right)^{1/n} \\
&= \lim_{n\to\infty} \left(\left|\frac{3^n}{n^3}\right|\right)^{1/n} = \lim_{n\to\infty} \frac{3}{n^{3/n}}
\end{aligned}$$

So we need to know what happens to $n^{3/n}$ as $n \to \infty$. This will turn out to require logarithms to solve.

$$\begin{aligned}
\lim_{n\to\infty} n^{3/n} &\longrightarrow \infty^0 \quad \text{indeterminate power} \\
y &= n^{3/n} \\
\ln y &= \ln n^{3/n} = \frac{3}{n}\ln n \\
\lim_{n\to\infty} \ln y &= \lim_{n\to\infty} \frac{3}{n}\ln n \longrightarrow \frac{\infty}{\infty} \quad \text{indeterminate quotient}
\end{aligned}$$

Now we should convert to the reals, since we want to use L'Hospital's Rule to evaluate this integral.

$$\begin{aligned}
\lim_{x\to\infty} \ln y &= 3\lim_{x\to\infty} \frac{\ln x}{x} \longrightarrow \frac{\infty}{\infty} \quad \text{indeterminate quotient} \\
&= \lim_{x\to\infty} \frac{1/x}{1} \quad \text{using L'Hospital's Rule} \\
&= 0
\end{aligned}$$

We want the limit

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e^0 = 1$$

So, since we had constructed the real function $x^{3/x}$ from the discrete $n^{3/n}$, we can also say

$$\lim_{n\to\infty} n^{3/n} = 1.$$

Therefore, we have

$$\lim_{n\to\infty} \left(|a_n|\right)^{1/n} = \lim_{n\to\infty} \frac{3}{n^{3/n}} = \frac{3}{1} = 3 > 1$$

so the series $\sum a_n$ diverges by the root test.

You could also use the ratio test to show the series diverges.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(-3)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(-3)^n}\right| \\
&= \lim_{n\to\infty} \frac{3n^3}{(n+1)^3} \\
&= 3\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 \\
&= 3\lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^3 = 3\left(\frac{1}{1+0}\right)^3 = 3 > 1
\end{aligned}$$

The series $\sum a_n$ diverges by the ratio test.

Example Is the series $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \frac{(-3)^n}{n!}$.

The a_n contains a factorial, so we should try the ratio test.

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-3)^n}\right| \\ &= \lim_{n\to\infty} \frac{3}{(n+1)} \\ &= 0 < 1 \end{aligned}$$

The series $\sum a_n$ is absolutely convergent by the ratio test.

Example Is the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{5+n}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \frac{(-1)^n}{5+n}$.

The a_n is alternating, so we should try the alternating series test.

For the alternating series test, we also need to identify $b_n = |a_n| = \frac{1}{5+n}$.

Since $b_{n+1} = \frac{1}{5+n+1} = \frac{1}{6+n} < \frac{1}{5+n} = b_n$ the first condition for the alternating series test is satisfied.

Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{5+n} = 0$, the second condition for the alternating series test is satisfied.

Therefore, by the alternating series test, the series $\sum a_n$ converges.

But we need to check the convergence of the series $\sum b_n$ to determine if the series $\sum a_n$ is conditionally convergent (that is, convergent due to the fact that it alternates).

Let's use the ratio test to check the series $\sum b_n$.

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{5+n}{6+n} \\ &= \lim_{n\to\infty} \frac{5/n+1}{6/n+1} \\ &= \frac{0+1}{0+1} \\ &= 1 \end{aligned}$$

so the ratio test fails. All this means is we can't use it.

Let's try a limit comparison test instead. Let's compare to the divergent p-series $\sum c_n = \sum 1/n$.

$$\lim_{n\to\infty} \frac{c_n}{b_n} = \lim_{n\to\infty} \frac{5+n}{n} = \lim_{n\to\infty} \left(\frac{5}{n} + 1\right) = 1 > 0 \text{ and finite.}$$

Therefore, the since the comparison series $\sum c_n$ was divergent, the series $\sum b_n$ is also divergent.

Therefore, $\sum a_n$ is conditionally convergent since $\sum a_n$ converges and $\sum |a_n| = \sum b_n$ diverges.

Example Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \dfrac{(-1)^{n-1}}{n!}$.

The a_n contains a factorial, so we should first try the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{\left(\frac{1}{(n+1)!}\right)}{\left(\frac{1}{n!}\right)} \\
&= \lim_{n\to\infty} \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{1}{n+1} \\
&= 0 < 1
\end{aligned}$$

so the series $\sum a_n$ is absolutely convergent by the ratio test.

Example Is the series $\displaystyle\sum_{n=1}^{\infty} e^{-n} n!$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = e^{-n} n!$.

The a_n contains a factorial, so we should first try the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{e^{-(n+1)}(n+1)!}{e^{-n} n!} \\
&= \lim_{n\to\infty} e^{-1}(n+1) \\
&= \frac{1}{e} \lim_{n\to\infty} (n+1) \\
&= \infty > 1
\end{aligned}$$

so the series $\sum a_n$ diverges by the ratio test.

Example Is the series $\displaystyle\sum_{n=2}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = (-1)^{n+1} \dfrac{n^2 2^n}{n!}$.

The a_n contains a factorial, so we should first try the ratio test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)^2 2^{n+1}}{(n+1)!}\cdot\frac{n!}{n^2 2^n} \\
&= \lim_{n\to\infty}\frac{2(n+1)^2}{(n+1)n^2} \\
&= 2\lim_{n\to\infty}\frac{n+1}{n^2} \\
&= 2\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right) \\
&= 2(0+0) = 0 < 1
\end{aligned}$$

so the series $\sum a_n$ is absolutely convergent by the ratio test.

Example Is the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \frac{(-1)^n}{n\ln n}$.

If we try the ratio test, it will fail.

Instead, since a_n is alternating, so we should try the alternating series test.

For the alternating series test, we also need to identify $b_n = |a_n| = \frac{1}{n\ln n}$.

Since $b_{n+1} = \frac{1}{(n+1)\ln(n+1)} < \frac{1}{n\ln n} = b_n$ the first condition for the alternating series test is satisfied.

Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n\ln n} = 0$, the second condition for the alternating series test is satisfied.

Therefore, by the alternating series test, the series $\sum a_n$ converges.

But we need to check the convergence of the series $\sum b_n$ to determine if the series $\sum a_n$ is conditionally convergent (that is, convergent due to the fact that it alternates).

Use the integral test to determine whether the series $\sum b_n$ is convergent or divergent. The integral test requires that we work with $f(x)$, where
1) $f(n) = a_n$,
and on the interval $[2,\infty)$, $f(x)$ is:
1) continuous,
2) positive,
3) decreasing.

Here, $f(x) = \frac{1}{x\ln x}$, which is continuous, decreasing and positive on the interval $[2,\infty)$.

We can therefore apply the integral test to the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

$$\begin{aligned}
\int_2^{\infty} f(x)\,dx &= \int_2^{\infty} \frac{1}{x \ln x}\,dx \\
&= \lim_{t\to\infty} \int_2^{t} \frac{1}{x \ln x}\,dx \\
&\text{Substitution: } \begin{array}{ll} u = \ln x & \text{when } x = 2, u = \ln 2 \\ du = \frac{1}{x}dx & \text{when } x = t, u = \ln t \end{array} \\
&= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u} \\
&= \lim_{t\to\infty} \ln u \Big|_{\ln 3}^{\ln t} \\
&= \frac{1}{2} \lim_{t\to\infty} (\ln \ln t - \ln \ln 2) \\
&= \infty, \quad \text{diverges, since } \ln \ln t \to \infty \text{ as } t \to \infty.
\end{aligned}$$

Since the integral diverges, the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the integral test. Therefore, the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

Therefore, $\sum a_n$ is conditionally convergent since $\sum a_n$ converges and $\sum |a_n| = \sum b_n$ diverges.

Example Is the series $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \left(\frac{n^2+1}{2n^2+1}\right)^n$.

The a_n contains a power involving n, so we should try the root test.

$$\begin{aligned}
\lim_{n\to\infty} (|a_n|)^{1/n} &= \lim_{n\to\infty} \left(\left|\left(\frac{n^2+1}{2n^2+1}\right)^n\right|\right)^{1/n} \\
&= \lim_{n\to\infty} \left(\left(\frac{n^2+1}{2n^2+1}\right)^n\right)^{1/n} \\
&= \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} \\
&= \lim_{n\to\infty} \frac{1+1/n^2}{2+1/n^2} \\
&= \frac{1+0}{2+0} \\
&= \frac{1}{2} < 1
\end{aligned}$$

so the series $\sum a_n$ is absolutely convergent by the root test.

Example Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x. Deduce that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x.

We identify $a_n = \dfrac{x^n}{n!}$.

The a_n contains a factorial, so we should first try the ratio test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| \\
&= |x|\lim_{n\to\infty}\frac{1}{n+1} \\
&= |x|(0) = 0 < 1
\end{aligned}$$

so the series $\sum a_n$ is absolutely convergent by the ratio test, for any value of x.

Since the series converges, it must be true that the terms in the series are approaching zero (by Theorem 11.2.6). Therefore, we know that for all values of x,

$$\lim_{n\to\infty}\frac{x^n}{n!} = 0.$$

This is an important result that we will use later. Check out Equation 11.10.10.