Questions

Example: The Alternating Harmonic Series Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converge or diverge?

Example Does the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converge or diverge?

Example Does the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$ converge or diverge?

Example How many terms are required to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

to 0.001 accuracy?

Example Test the series for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}.$$

Solutions

Example: The Alternating Harmonic Series Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converge or diverge?

Since this is an alternating series, we should use the alternating series test. First, we identify

$$a_n = (-1)^{n-1} \frac{1}{n}, \quad b_n = |a_n| = \frac{1}{n}.$$

Since $1/(n+1) < 1/n$, we have that $b_{n+1} < b_n$, so the condition $b_{n+1} \leq b_n$ is satisfied.

Secondly, we have that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/n = 0$.

So the two conditions of the alternating series test are satisfied, and the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges.

NOTE: The series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n$ is divergent. We can prove this by the integral test. This isn't asked for in this problem, but let's prove it, because it is fun!

The integral test requires that we work with $f(x)$, where
1) $f(n) = b_n$,
and on the interval $[1, \infty)$, $f(x)$ must be:

1) continuous,
2) positive,
3) decreasing.

So $f(x) = 1/x$, which is continuous, positive, and decreasing on $[1, \infty)$.

$$\begin{aligned}
\int_1^\infty f(x)\,dx &= \int_1^\infty \frac{1}{x}\,dx \\
&= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx \\
&= \lim_{b\to\infty} \ln x|_1^b \\
&= \lim_{b\to\infty} (\ln b - \ln 1) \\
&= \lim_{b\to\infty} \ln b \\
&= \infty
\end{aligned}$$

So the integral diverges. Therefore, $\sum_{n=1}^\infty 1/n$ diverges by the integral theorem.

A series which has $\sum a_n$ convergent, but $\sum |a_n|$ divergent is called *conditionally convergent*, since the convergence is due to the cancellation that occurs by change in sign of the terms. This is a property we will look at more deeply in Section 11.6.

Example Does the series $\displaystyle\sum_{n=2}^\infty \frac{(-1)^n}{\ln n}$ converge or diverge?

Since this is an alternating series, we should use the alternating series test. First, we identify

$$a_n = (-1)^n \frac{1}{\ln n}, \quad b_n = |a_n| = \frac{1}{\ln n}.$$

Since $\ln(n+1) > \ln n \longrightarrow 1/(\ln(n+1)) < 1/\ln n$, we have $b_{n+1} < b_n$, so the condition $b_{n+1} \le b_n$ is satisfied.

Secondly, we have that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/\ln n = 0$.

So the two conditions of the alternating series test are satisfied, and the series $\displaystyle\sum_{n=2}^\infty \frac{(-1)^n}{\ln n}$ converges.

Example (11.5.14) Does the series $\displaystyle\sum_{n=1}^\infty (-1)^{n-1} \frac{\ln n}{n}$ converge or diverge?

Since this is an alternating series, we should use the alternating series test. First, we identify

$$a_n = (-1)^{n-1} \frac{\ln n}{n}, \quad b_n = |a_n| = \frac{\ln n}{n}.$$

We need to show that $b_{n+1} < b_n$, which is not obvious from what we have for b_n, since both the numerator and denominator are increasing.

However, if a function $f(x)$ is decreasing, then it must be true that $f'(x) < 0$. If we can show that $f(x) = \dfrac{\ln x}{x}$ has $f'(x) < 0$, then since $f(n) = b_n$, we will have shown that $b_{n+1} < b_n$.

Let's take the derivative of $f(x)$:

$$\frac{d}{dx} f(x) = \frac{d}{dx} \frac{\ln x}{x} = \frac{1 - \ln x}{x^2}$$

For this to be less than zero, we require $1 - \ln x < 0 \longrightarrow x > e$. This will certainly be true if $x > 3$, since $e \sim 2.71828$.

Therefore, $b_{n+1} < b_n$ for $n = 3, 4, 5, \ldots$, so the terms are eventually decreasing.

Also, we need to show that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} = 0$. Again, this isn't obvious, since direct substitution leads to an indeterminant quotient. We will have to switch to the continuous counterpart so we can use l'Hospital's rule.

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{\ln x}{x} \to \frac{\infty}{\infty} \quad \text{indeterminant quotient, use l'Hospital's rule} \\ &= \lim_{x\to\infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{1}{x} = 0 \end{aligned}$$

Therefore, since $\lim_{x\to\infty} f(x) = 0$, and $f(n) = b_n$, we have $\lim_{n\to\infty} b_n = 0$.

We have shown that for $b_n = |a_n|$, $\lim_{n\to\infty} b_n = 0$ and $b_{n+1} \le b_n$, so the series $\sum a_n$ converges by the alternating series test.

Example How many terms are required to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

to 0.001 accuracy?

First, we have to check that the series converges by the alternating series test. Then we can use the remainder estimate for the alternating series test.

Here, we have

$$a_n = \frac{(-1)^{n+1}}{n^4}, \quad b_n = \frac{1}{n^4}.$$

Since $1/(n+1)^4 < 1/n^4$, we have that $b_{n+1} < b_n$, so the condition $b_{n+1} \le b_n$ is satisfied.

Secondly, we have that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/n^4 = 0$.

So the two conditions of the alternating series test are satisfied, and the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}$ converges.

The remainder estimate for the alternating series test tells us that if we approximate the series sum s by the partial sum s_n, the error will be

$$|R_n| \leq b_{n+1}.$$

n	b_n
1	1.0
2	0.0625
3	0.0123457
4	0.003906
5	0.0016
6	0.00077

Since $b_6 < 0.001$, we can say that

$$\begin{aligned} |R_n| &\leq b_{n+1} \\ |R_5| &\leq b_6 = 0.00077 \end{aligned}$$

So using the first five terms will produce an accuracy of 0.001.

Example Test the series for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}.$$

Although this doesn't initially look like an alternating series, it is an alternating series since the sine function alternates

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!} = 1 + 0 - \frac{1}{6} + 0 + \frac{1}{120} + 0 - \frac{1}{5040} + \ldots = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \ldots.$$

We therefore have

$$a_n = \frac{\sin(n\pi/2)}{n!}$$

and since

$$\begin{aligned} \sum_{i=1}^{\infty} b_n &= \sum_{i=1}^{\infty} \left| \frac{\sin(n\pi/2)}{n!} \right| = 1 + \frac{1}{6} + \frac{1}{120} + \frac{1}{5040} + \ldots \\ &= \sum_{i=1}^{\infty} \frac{1}{(2n-1)!} \\ \longrightarrow b_n &= \frac{1}{(2n-1)!} \end{aligned}$$

Since $\frac{1}{(2(n+1)-1)!} = \frac{1}{(2n+1)!} < \frac{1}{(2n-1)!}$, we have that $b_{n+1} < b_n$, so the condition $b_{n+1} \leq b_n$ is satisfied.

Secondly, we have that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/(2n-1)! = 0$.

So the two conditions of the alternating series test are satisfied, and the series $\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}$ converges.