Questions

Example Is $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ convergent or divergent?

Example Is $\sum_{n=1}^{\infty} \frac{2n^2+7n}{3^n(n^2+5n-1)}$ convergent or divergent?

Example Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to be convergent.

(a) If $a_n > b_n$ for all n, what can you say about $\sum a_n$? Why?
(b) If $a_n < b_n$ for all n, what can you say about $\sum a_n$? Why?

Example Determine whether the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$.

Example Is $\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$ convergent or divergent?

Example Determine whether the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$.

Example Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is convergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

then $\sum a_n$ is convergent.

Note: This is a more difficult problem than most, since it is a proof that involves the definition of limit.

Example Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is divergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

then $\sum a_n$ is divergent.

Note: This is a more complicated problem than most, and involves using a *proof by contradiction*.

Solutions

Example Is $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ convergent or divergent?

Let's try to use the Comparison Test. How do we know what series to compare to? Well, we try something, and use a series which we know something about. We usually try to pick our comparison series based on attributes of the given series.

$$\begin{aligned} \ln n &> 1 \quad \text{for } n \geq 3 \\ a_n = \frac{\ln n}{n} &> \frac{1}{n} = b_n \quad \text{for } n \geq 3 \end{aligned}$$

Since $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ is divergent (it is a p-series with $p = 1$), we know $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent by the comparison test.

Example Is $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 7n}{3^n(n^2 + 5n - 1)}$ convergent or divergent?

Let's try to use the Limit Comparison Test. For large n,

$$2n^2 + 7n \sim 2n^2$$

$$3^n(n^2 + 5n - 1) \sim 3^n n^2$$

So let's take

$$a_n = \frac{2n^2 + 7n}{3^n(n^2 + 5n - 1)} \qquad b_n = \frac{2n^2}{3^n n^2} = \frac{2}{3^n}$$

The series $\displaystyle\sum b_n = \sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{n-1}$ is a convergent geometric series (since $a = 2/3$, $|r| = 1/3 < 1$).

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\left(\dfrac{2n^2 + 7n}{3^n(n^2 + 5n - 1)}\right)}{\left(\dfrac{2}{3^n}\right)} \\ &= \lim_{n\to\infty} \left(\frac{2n^2 + 7n}{3^n(n^2 + 5n - 1)}\right)\left(\frac{3^n}{2}\right) \\ &= \lim_{n\to\infty} \left(\frac{2n^2 + 7n}{2(n^2 + 5n - 1)}\right) \\ &= \lim_{n\to\infty} \left(\frac{2 + 7/n}{2(1 + 5/n - 1/n^2)}\right) \\ &= 1 > 0 \text{ and finite.} \end{aligned}$$

Since $\sum b_n$ converges, $\sum a_n$ converges by the limit comparison test.

Example Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to be convergent.

(a) If $a_n > b_n$ for all n, what can you say about $\sum a_n$? Why?
(b) If $a_n < b_n$ for all n, what can you say about $\sum a_n$? Why?

(a) If $a_n > b_n$ for all n, and $\sum b_n$ is convergent, then we cannot say anything about $\sum a_n$ since it is not bounded above by $\sum b_n$.

(b) Since a_n is positive, the series $\sum a_n$ must be increasing since we are always adding a positive quantity to the partial sum (in other words, $s_{n+1} > s_n$). If $a_n < b_n$ for all n, and $\sum b_n$ is convergent, then $\sum a_n$ is convergent since it is bounded above by $\sum b_n$ which converges.

Example (11.4.3) Determine whether the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$.

Let's try to use the Comparison Test. Let's try to pick our comparison series based on attributes of the given series.

$$\begin{aligned} n^2+n+1 &> n^2 \quad \text{for } n \geq 1 \\ a_n = \frac{1}{n^2+n+1} &< \frac{1}{n^2} = b_n \quad \text{for } n \geq 1 \quad \text{(note change in the relation)} \end{aligned}$$

Since $\displaystyle\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (it is a p-series with $p = 2$), we know $\displaystyle\sum a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ is convergent by the comparison test.

Example Is $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$ convergent or divergent?

Let's try to use the Limit Comparison Test. For large n,

$$2^n(n+1) \sim 2^n n$$

So let's take

$$a_n = \frac{n}{2^n(n+1)} \qquad b_n = \frac{n}{2^n n} = \frac{1}{2^n}$$

The series $\displaystyle\sum b_n = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$ is a convergent geometric series (since $a = 1/2$, $|r| = 1/2 < 1$).

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\left(\dfrac{n}{2^n(n+1)}\right)}{\left(\dfrac{1}{2^n}\right)} \\ &= \lim_{n\to\infty} \left(\frac{n}{2^n(n+1)}\right)(2^n) \\ &= \lim_{n\to\infty} \left(\frac{n}{n+1}\right) \\ &= \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right) \\ &= 1 > 0 \text{ and finite.} \end{aligned}$$

Since $\sum b_n$ converges, $\sum a_n$ converges by the limit comparison test.

Example Determine whether the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$.

Let's try to use the Comparison Test. Let's try to pick our comparison series based on attributes of the given series.

$$\begin{aligned} 1+\sqrt{n} &> \sqrt{n} \quad \text{for } n \geq 1 \\ a_n = \frac{1}{1+\sqrt{n}} &< \frac{1}{\sqrt{n}} = b_n \quad \text{for } n \geq 1 \quad \text{(note change in the relation)} \end{aligned}$$

Since $\displaystyle\sum b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent (it is a p-series with $p = 1/2$), this doesn't tell us anything about $\sum a_n$ (see (11.1.1)). Since this doesn't help us, we'll have to try something else.

Let's try the limit comparison test with the comparison series $\displaystyle\sum b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\left(\dfrac{1}{1+\sqrt{n}}\right)}{\left(\dfrac{1}{\sqrt{n}}\right)} \\ &= \lim_{n\to\infty} \left(\frac{1}{1+\sqrt{n}}\right)(\sqrt{n}) \\ &= \lim_{n\to\infty} \left(\frac{\sqrt{n}}{1+\sqrt{n}}\right) \\ &= \lim_{n\to\infty} \left(\frac{1}{1+1/\sqrt{n}}\right) \\ &= 1 > 0 \text{ and finite.} \end{aligned}$$

Since $\sum b_n$ diverges, $\sum a_n$ diverges by the limit comparison test.

Example Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is convergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

then $\sum a_n$ is convergent.

Note: This is a more difficult problem than most, since it is a proof that involves the definition of limit.

Since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, by the definition of limit we know there exists an $N > 0$ such that $|a_n/b_n - 0| < 1$ for all $n > N$.

Since a_n and b_n are positive, $|a_n/b_n - 0| < 1 \longrightarrow a_n < b_n$.

Therefore, since $\sum b_n$ converges, $\sum a_n$ converges by the comparison test.

Example Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is divergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

then $\sum a_n$ is divergent.

Note: This is a more complicated problem than most, and involves using a *proof by contradiction*.

Assume $\sum a_n$ converges.

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \longrightarrow \lim_{n\to\infty} \frac{b_n}{a_n} = 0$.

Using the result from Problem (11.4.40 a), we know that if $\sum a_n$ converges then $\sum b_n$ converges as well.

But we are told that $\sum b_n$ diverges (contradiction). Therefore, the assumption we made must be wrong, and $\sum a_n$ diverges.