Questions

Example (The p series) Get upper and lower bounds on the sum for the p series $\sum_{i=1}^{\infty} 1/i^p$ with $p = 2$ if the 4th partial sum is used to estimate the sum.

Example Use the integral test to determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Example Use the integral test to determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1}{3n+1}.$$

Example Is the following series divergent or convergent?

$$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$$

Example Is the following series divergent or convergent? $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Note: this is one of the more complex problems that can arise that uses the integral test.

Solutions

Example (The p series) Get upper and lower bounds on the sum for the p series $\sum_{i=1}^{\infty} 1/i^p$ with $p = 2$ if the 4th partial sum is used to estimate the sum.

Since the p series for $p = 2$ can be shown to converge using the integral test (see Example 2), we want to use the result that

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_{n}^{\infty} f(x)\,dx$$

which for the 4th partial sum is written as

$$s_4 + \int_{5}^{\infty} f(x)\,dx \le s \le s_4 + \int_{4}^{\infty} f(x)\,dx \tag{1}$$

where $f(i) = 1/i^2$.

First, we need the 4th partial sum:

$$s_4 = \sum_{i=1}^{4} \frac{1}{i^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{205}{144}$$

Now we need our integral estimates for the remainder. Let's work with a general lower limit, since that is the only thing that changes in the two integrals we must perform.

$$\begin{aligned}
\int_a^\infty f(x)\,dx &= \int_a^\infty \frac{1}{x^2}\,dx \\
&= \lim_{b\to\infty} \int_a^b x^{-2}\,dx \\
&= \lim_{b\to\infty} \left.\frac{x^{-1}}{(-1)}\right|_a^b \\
&= -\lim_{b\to\infty} \left.\frac{1}{x}\right|_a^b \\
&= -\lim_{b\to\infty} \left(\frac{1}{b} - \frac{1}{a}\right) = \frac{1}{a} \\
\int_4^\infty f(x)\,dx &= \frac{1}{4} \\
\int_5^\infty f(x)\,dx &= \frac{1}{5}
\end{aligned}$$

Substituting into Eq. (1) above, we have

$$\frac{205}{144} + \frac{1}{5} \le s \le \frac{205}{144} + \frac{1}{4} \quad \text{or} \quad 1.62 \le s \le 1.67$$

Example Use the integral test to determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

The integral test requires that we work with $f(x)$, where
1) $f(n) = a_n$,
and on the interval $[1, \infty)$, $f(x)$ is:
1) continuous,
2) positive,
3) decreasing.

Here, $f(x) = x^{-4}$, which is continuous, positive, and decreasing on the interval $[1, \infty)$. So the integral test can be used.

$$\int_1^\infty f(x)\,dx = \int_1^\infty x^{-4}\,dx$$

$$\begin{aligned}
&= \lim_{t\to\infty}\int_1^t x^{-4}\,dx \\
&= -\lim_{t\to\infty}\frac{1}{3}x^{-3}\Big|_1^t \\
&= -\frac{1}{3}\lim_{t\to\infty}\left(\frac{1}{t^3}-1\right) \\
&= -\frac{1}{3}(0-1) = \frac{1}{3}
\end{aligned}$$

Since the integral converges, the series $\sum_{n=1}^{\infty}\frac{1}{n^4}$ converges by the integral test. Note that $\sum_{n=1}^{\infty}\frac{1}{n^4} \neq \frac{1}{3}$!

Example Use the integral test to determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty}\frac{1}{3n+1}.$$

The integral test requires that we work with $f(x)$, where
1) $f(n) = a_n$,
and on the interval $[1,\infty)$, $f(x)$ is:
1) continuous,
2) positive,
3) decreasing.

Here, $f(x) = (3x+1)^{-1}$, which is continuous, positive, and decreasing on the interval $[1,\infty)$. So the integral test can be used.

$$\begin{aligned}
\int_1^{\infty} f(x)\,dx &= \int_1^{\infty}(3x+1)^{-1}\,dx \\
&= \lim_{t\to\infty}\int_1^t \frac{1}{3x+1}\,dx \\
&\quad \text{Substitution: } \begin{array}{ll} u = 3x+1 & \text{when } x=1, u=4 \\ du = 3\,dx & \text{when } x=t, u=3t+1 \end{array} \\
&= \frac{1}{3}\lim_{t\to\infty}\int_4^{3t+1}\frac{du}{u} \\
&= \frac{1}{3}\lim_{t\to\infty}(\ln|u|)_4^{3t+1} \\
&= \frac{1}{3}\lim_{t\to\infty}(\ln|3t+1| - \ln 4) \\
&= \infty, \quad \text{diverges, since } \ln x \to \infty \text{ as } x \to \infty.
\end{aligned}$$

Since the integral diverges, the series $\sum_{n=1}^{\infty}\frac{1}{3n+1}$ diverges by the integral test.

Example Is the following series divergent or convergent?

$$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$$

The series can be rewritten as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Use the integral test to determine whether the series is convergent or divergent. The integral test requires that we work with $f(x)$, where
1) $f(n) = a_n$,
and on the interval $[1, \infty)$, $f(x)$ is:
1) continuous,
2) positive,
3) decreasing.

Here, $f(x) = x^{-3}$, which is continuous, positive, and decreasing on the interval $[1, \infty)$. So the integral test can be used.

$$\begin{aligned}
\int_1^{\infty} f(x)\,dx &= \int_1^{\infty} x^{-3}\,dx \\
&= \lim_{t\to\infty} \int_1^t x^{-3}\,dx \\
&= -\lim_{t\to\infty} \frac{1}{2}x^{-2}\Big|_1^t \\
&= -\frac{1}{2}\lim_{t\to\infty}\left(\frac{1}{t^2} - 1\right) \\
&= -\frac{1}{2}(0-1) = \frac{1}{2}
\end{aligned}$$

Since the integral converges, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the integral test. Note that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} \neq \frac{1}{2}$!

Example Is the following series divergent or convergent? $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Note: this is one of the more complex problems that can arise that uses the integral test.

Use the integral test to determine whether the series is convergent or divergent. The integral test requires that we work with $f(x)$, where
1) $f(n) = a_n$,
and on the interval $[1, \infty)$, $f(x)$ is:
1) continuous,
2) positive,
3) decreasing.

Here, $f(x) = \dfrac{\ln x}{x}$, which is continuous and positive on the interval $[1, \infty)$.

But is it decreasing on this interval? It is not obvious, since both the numerator and denominator are increasing functions of x.

However, if a function $f(x)$ is decreasing, then it must be true that $f'(x) < 0$. Let's take the derivative of $f(x)$ and see what we can learn.

$$\frac{d}{dx}f(x) = \frac{d}{dx}\frac{\ln x}{x} = \frac{1-\ln x}{x^2}$$

For this to be less than zero, we require $1 - \ln x < 0 \longrightarrow x > e$. This will certainly be true if $x > 3$, since $e \sim 2.71828$.

We can therefore apply the integral test to the series $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$. Note that we start at $n = 3$ and not $n = 1$, since we must work on the interval $[3, \infty)$.

$$
\begin{aligned}
\int_3^\infty f(x)\,dx &= \int_3^\infty \frac{\ln x}{x}\,dx \\
&= \lim_{t\to\infty} \int_3^t \frac{\ln x}{x}\,dx \\
& \qquad \text{Substitution: } \begin{array}{ll} u = \ln x & \text{when } x = 3, u = \ln 3 \\ du = \frac{1}{x}dx & \text{when } x = t, u = \ln t \end{array} \\
&= \lim_{t\to\infty} \int_{\ln 3}^{\ln t} u\,du \\
&= \frac{1}{2} \lim_{t\to\infty} u^2 \Big|_{\ln 3}^{\ln t} \\
&= \frac{1}{2} \lim_{t\to\infty} \left(\ln^2 t - \ln^2 3\right) \\
&= \infty, \quad \text{diverges, since } \ln^2 t \to \infty \text{ as } t \to \infty.
\end{aligned}
$$

Since the integral diverges, the series $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges by the integral test. Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.