Example Find the Taylor series of $f(x) = e^{-x}$ about $x = 0$.

The center of our Taylor series will be $a = 0$. This means it could be called a MacLaurin series.

Let's construct a table which will give us the derivatives, and enable us to calculate $f^{(n)}(a)$. We will want the general form, so we should try and write things in ways in which the pattern becomes evident.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(0)$
0	e^{-x}	1
1	$-e^{-x}$	-1
2	$+e^{-x}$	$+1$
3	$-e^{-x}$	-1
$\vdots$	$\vdots$	$\vdots$
n	$(-1)^n e^{-x}$	$(-1)^n$

So we can see that the general form is $f^{(n)}(0) = (-1)^n$, since if we take an even derivative we get a positive number, and if we take an odd derivative the number is negative.

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n}{n!}$$

The Taylor series is given by

$$e^{-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n, \quad |x| < R.$$

Now we want to find the radius of convergence, R. We can do this using the ratio test, where $a_n = \dfrac{(-1)^n}{n!} x^n$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^n} \right| \\
&= \lim_{n\to\infty} \left| x \frac{n!}{(n+1)!} \right| \\
&= |x| \lim_{n\to\infty} \frac{1}{n+1} \\
&= |x| \cdot 0 = 0 < 1 \quad \text{for all } x.
\end{aligned}$$

So the series is absolutely convergent for all values of x, which means $R = \infty$.

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n, \quad x \in (-\infty, \infty).$$

This can be checked in *Mathematica* using:

```
f[x_] = Exp[-x]
Series[f[x], {x, 0, 5}]
```

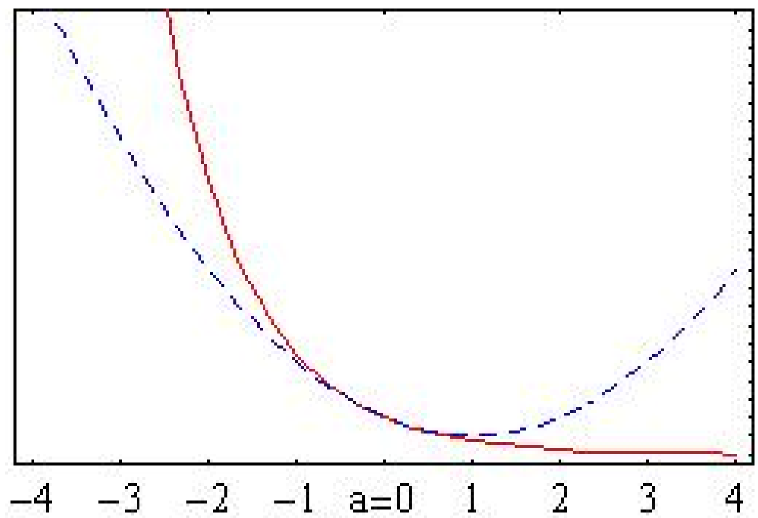

Figure 1: Plots of $f(x) = e^{-x}$ (red) and the Taylor polynomial approximation of order 2 centered at $a = 2$, $T_2(x) = \sum_{n=0}^{2} \frac{(-1)^n}{n!} x^n = 1 - x + \frac{x^2}{2}$ (blue).

Example Find the Taylor series of $f(x) = e^{-x}$ about $x = 3$.

The center of our Taylor series will be $a = 3$.

Let's construct a table which will give us the derivatives, and enable us to calculate $f^{(n)}(a)$.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(3)$
0	e^{-x}	$1e^{-3}$
1	$-e^{-x}$	$-1e^{-3}$
2	$+e^{-x}$	$+1e^{-3}$
3	$-e^{-x}$	$-1e^{-3}$
$\vdots$	$\vdots$	$\vdots$
n	$(-1)^n e^{-x}$	$(-1)^n e^{-3}$

So we can see that the general form is $f^{(n)}(3) = (-1)^n e^{-3}$.

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n}{e^3 n!}$$

The Taylor series is given by

$$e^{-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{e^3 n!}(x-3)^n, \quad |x-3| < R.$$

Now we want to find the radius of convergence, R. We can do this using the ratio test, where

$a_n = \dfrac{(-1)^n}{e^3 n!}(x-3)^n.$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-3)^{n+1}}{e^3(n+1)!}\cdot\frac{e^3 n!}{(-1)^n(x-3)^n}\right| \\ &= \lim_{n\to\infty}\left|(x-3)\frac{n!}{(n+1)!}\right| \\ &= |x-3|\lim_{n\to\infty}\frac{1}{n+1} \\ &= |x-3|\cdot 0 = 0 < 1 \quad \text{for all } x.\end{aligned}$$

So the series is absolutely convergent for all values of x, which means $R = \infty$.

$$e^{-x} = \sum_{n=0}^{\infty}\frac{(-1)^n}{e^3 n!}(x-3)^n, \quad x \in (-\infty, \infty).$$

This can be checked in *Mathematica* using:

```
f[x_] = Exp[-x]
Series[f[x], {x, 3, 5}]
```

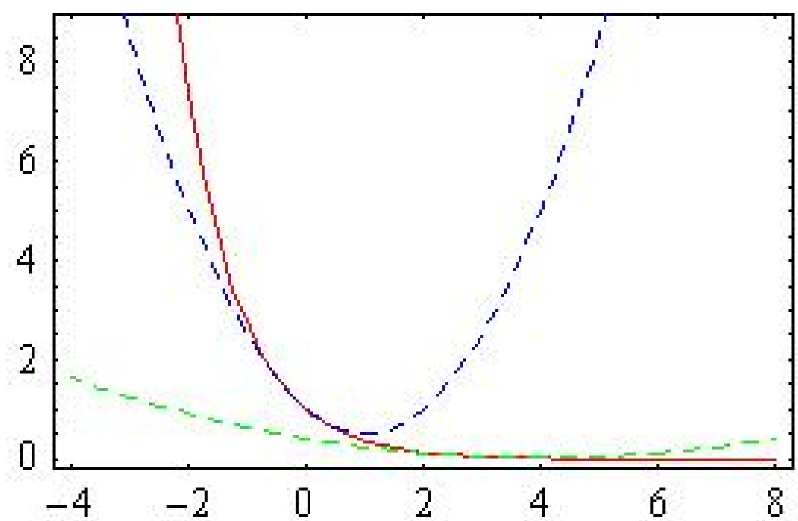

Figure 2: Plots of $f(x) = e^{-x}$ (red) and the two Taylor polynomial approximations of order 2, one centered at $a = 0$ (blue) and the other centered at $a = 3$ (green).

Example 11.10.12 Find the Taylor series of $f(x) = \ln x$ about $x = 2$.

The center of our Taylor series will be $a = 2$.

Let's construct a table which will give us the derivatives, and enable us to calculate $f^{(n)}(a)$.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$1/2$
2	$-x^{-2}$	$-1/2^2$
3	$+2x^{-3}$	$+2/2^3$
4	$-2 \cdot 3x^{-4}$	$-2 \cdot 3/2^4$
$\vdots$	$\vdots$	$\vdots$
$n \neq 0$	$(-1)^{n+1}(n-1)!\frac{1}{x^n}$	$(-1)^{n+1}(n-1)!\frac{1}{2^n}$

So we can see that the general form is $f^{(n)}(2) = (-1)^{n+1}(n-1)!\frac{1}{2^n}$ if $n \neq 0$, and $f^{(0)}(2) = \ln 2$. Since the form changes, we will have to pull the $n = 0$ term out of our sum.

$$c_n = \frac{f^{(n)}(2)}{n!} = \frac{(-1)^{n+1}(n-1)!\frac{1}{2^n}}{n!} = \frac{(-1)^{n+1}\frac{1}{2^n}}{n}, \ \ n \neq 0; \quad c_0 = \ln 2$$

The Taylor series is given by

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n n}(x-2)^n, \quad |x-2| < R.$$

Now we want to find the radius of convergence, R. We can do this using the ratio test, where $a_n = \dfrac{(-1)^{n+1}}{2^n n}(x-2)^n$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1}}{2^{n+1}(n+1)} \cdot \frac{2^n n}{(-1)^{n+1}(x-2)^n}\right| \\
&= \lim_{n\to\infty}\left|(x-2)^{n+1-n}2^{n-n-1} \cdot \frac{n}{n+1}\right| \\
&= \frac{|x-2|}{2}\lim_{n\to\infty}\frac{n}{n+1} \\
&= \frac{|x-2|}{2}\lim_{n\to\infty}\frac{1}{1+1/n} \\
&= \frac{|x-2|}{2} \cdot \frac{1}{1+0} = \frac{|x-2|}{2} < 1
\end{aligned}$$

So the series is absolutely convergent for $|x-2| < 2$ which means $R = 2$.

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n n}(x-2)^n, \quad |x-2| < 2.$$

This can be checked in *Mathematica* using:

```
f[x_] = Log[x]
Series[f[x], {x, 2, 5}]
```

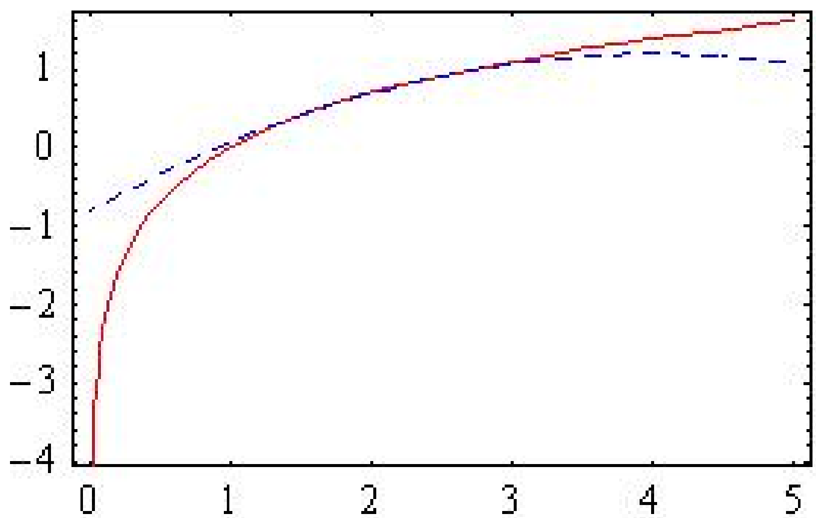

Figure 3: Plots of $f(x) = \ln x$ (red) and the Taylor polynomial approximation of order 4 centered at $a = 2$, $T_4(x) = \ln 2 + \sum_{n=1}^{4} \frac{(-1)^{n+1}}{2^n n}(x-2)^n$ (blue).

Example 11.11.2 Find the Taylor series of $f(x) = 1/(1+x)^4$ about $x = 0$.

The center of our Taylor series will be $a = 0$.

Let's construct a table which will give us the derivatives, and enable us to calculate $f^{(n)}(a)$.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(0)$
0	$(1+x)^{-4}$	1
1	$-4(1+x)^{-5}$	-4
2	$4 \cdot 5\,(1+x)^{-6}$	$+4 \cdot 5$
3	$-4 \cdot 5 \cdot 6\,(1+x)^{-7}$	$-4 \cdot 5 \cdot 6$
4	$4 \cdot 5 \cdot 6 \cdot 7\,(1+x)^{-8}$	$+4 \cdot 5 \cdot 6 \cdot 7$
$\vdots$	$\vdots$	$\vdots$
n	$(-1)^n \frac{1}{2 \cdot 3}(n+3)!(1+x)^{-(n+4)}$	$(-1)^n \frac{(n+3)!}{6}$

So we can see that the general form is $f^{(n)}(0) = (-1)^n \frac{(n+3)!}{6}$.

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n \frac{(n+3)!}{6}}{n!} = (-1)^n \frac{(n+1)(n+2)(n+3)}{6}.$$

The Taylor series is given by

$$\frac{1}{(1+x)^4} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)(n+3)}{6} x^n, \quad |x| < R.$$

Now we want to find the radius of convergence, R. We can do this using the ratio test, where

$a_n = (-1)^n \dfrac{(n+1)(n+2)(n+3)}{6} x^n.$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)(n+3)(n+4)x^{n+1}}{6}\cdot\frac{6}{(-1)^n x^n (n+1)(n+2)(n+3)}\right| \\
&= \lim_{n\to\infty}\left|x^{n+1-n}\cdot\frac{(n+2)(n+3)(n+4)}{(n+1)(n+2)(n+3)}\right| \\
&= |x|\lim_{n\to\infty}\left|\frac{(n+4)}{(n+1)}\right| \\
&= |x|\lim_{n\to\infty}\left|\frac{(1+4/n)}{(1+1/n)}\right| \\
&= |x|\cdot\frac{1+0}{1+0} = |x| < 1
\end{aligned}$$

So the series is absolutely convergent for $|x| < 1$ which means $R = 1$.

$$\frac{1}{(1+x)^4} = \sum_{n=0}^{\infty}(-1)^n\frac{(n+1)(n+2)(n+3)}{6}x^n, \quad |x| < 1.$$

```
f[x_] = 1/(1+x)^4
Series[f[x], {x, 0, 5}]
```

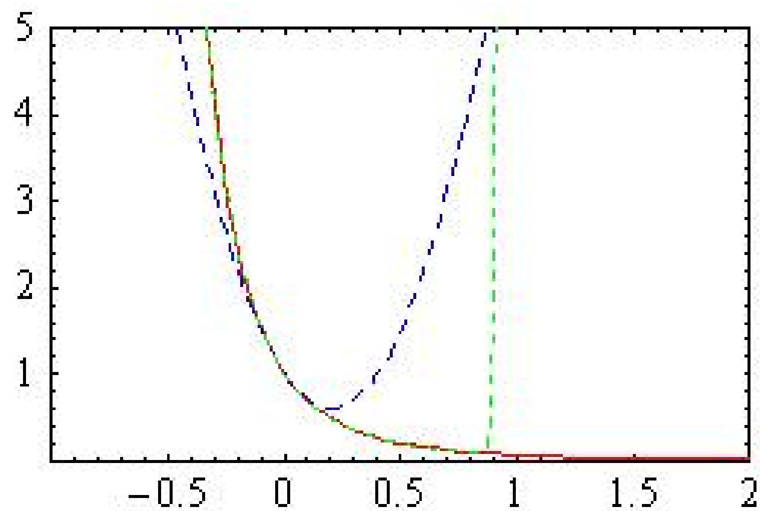

Figure 4: Plots of $f(x) = 1/(1+x)^4$ (red) and the Taylor polynomial approximation centered at $a = 0$ of order 2 (blue), and 100 (green).

Example 11.10.35 Find the MacLaurin series for $f(x) = \ln(1+x)$ and use it to calculate $\ln 1.1$ to five decimals.

The center of our Taylor series will be $a = 0$.

Let's construct a table which will give us the derivatives, and enable us to calculate $f^{(n)}(a)$.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(0)$
0	$\ln(1+x)$	$\ln 1 = 0$
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$+2(1+x)^{-3}$	$+2$
4	$-2 \cdot 3(1+x)^{-4}$	$-2 \cdot 3$
$\vdots$	$\vdots$	$\vdots$
$n \neq 0$	$(-1)^{n+1}(n-1)!\dfrac{1}{(1+x)^n}$	$(-1)^{n+1}(n-1)!$

So we can see that the general form is $f^{(n)}(0) = (-1)^{n+1}(n-1)!$ if $n \neq 0$, and $f^{(0)}(0) = 0$. Since the form changes, we will have to pull the $n = 0$ term out of our sum.

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}(n-1)!}{n!} = \frac{(-1)^{n+1}}{n}, \;\; n \neq 0; \quad c_0 = 0$$

The Taylor series is given by

$$\ln(1+x) = 0 + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \quad |x| < R.$$

Now we want to find the radius of convergence, R. We can do this using the ratio test, where $a_n = \dfrac{(-1)^{n+1}}{n} x^n$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+2}x^{n+1}}{(n+1)} \cdot \frac{n}{(-1)^{n+1}x^n} \right| \\
&= \lim_{n\to\infty} \left| x^{n+1-n} \cdot \frac{n}{n+1} \right| \\
&= |x| \lim_{n\to\infty} \frac{n}{n+1} \\
&= |x| \lim_{n\to\infty} \frac{1}{1+1/n} \\
&= |x| \cdot \frac{1}{1+0} = |x| < 1
\end{aligned}$$

So the series is absolutely convergent for $|x| < 1$ which means $R = 1$.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \quad |x| < 1.$$

We can use this to get an estimate for $\ln 1.1$:

$$\begin{aligned}
\ln 1.1 = \ln(1+0.1) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(0.1)^n \\
&= 0.1 - 0.005 + 0.0003333 - 0.000025 + 0.000002 - \cdots
\end{aligned}$$

Since the series is alternating, we can estimate the error in truncating by the first term dropped, $|R_n| \le |a_{n+1}|$. We can therefore write

$$\begin{aligned} \ln 1.1 &= 0.1 - 0.005 + 0.0003333 - 0.000025 + 0.000002 - \cdots \\ &\sim 0.1 - 0.005 + 0.0003333 - 0.000025, \quad |\text{error}| \le 0.000002 \\ &\sim 0.09531, \quad |\text{error}| \le 0.000002 \end{aligned}$$

I used the *Mathematica* commands:

```
a[n_] = (-1)^(n + 1)/n (0.1)^n
a[1]
a[2]
a[3]
a[4]
a[5]
a[1] + a[2] + a[3] + a[4]
```

and to check the answer I used

```
f[x_] = Log[1 + x]
g[x_] = Normal[Series[f[x], {x, 0, 5}]]
Log[1.1]
```

Example 11.10.44 Use series to approximate the definite integral to within an $|\text{error}| < 0.001$

$$\int_0^{1/2} x^2 e^{-x^2}\, dx.$$

We need to expand the integrand as a series. However, it is complicated looking; we might not be able to find a pattern.

Let's use *Mathematica* to help us get the derivatives we need to form the Taylor series. The problem doesn't tell us what to use as the center; I choose to use $a = 0$, although other values for the center will work.

```
f[x_] = x^2 Exp[-x^2]
Simplify[f'[x]]
Simplify[f''[x]]
Simplify[f'''[x]]
Simplify[f''''[x]]
```

The derivatives look kind of complicated. Here they are:

$$f(x) = x^2 e^{-x^2}$$

$$f^{(1)}(x) = -2xe^{-x^2}(-1 + x^2)$$

$$f^{(2)}(x) = 2e^{-x^2}(1 - 5x^2 + 2x^4)$$

$$f^{(3)}(x) = -4e^{-x^2}x(6 - 9x^2 + 2x^4)$$

$$f^{(4)}(x) = 4e^{-x^2}(-6 + 39x^2 - 28x^4 + 4x^6)$$

I evaluated them at $x = a = 0$ and found:

```
Simplify[f[0]]
Simplify[f'[0]]
Simplify[f''[0]]
Simplify[f'''[0]]
Simplify[f''''[0]]
```

$$f(x) = 0$$

$$f^{(1)}(x) = 0$$

$$f^{(2)}(x) = 2$$

$$f^{(3)}(x) = 0$$

$$f^{(4)}(x) = -24$$

Wow! Lot's of zeros! So we have for our Taylor series

$$f(x) = x^2e^{-x^2} \sim 0 + 0 + \frac{2}{2!}x^2 + 0 - \frac{24}{4!}x^4 = x^2 - x^4$$

This simplified version looks like there might be a simple pattern after all. Let's get some more terms and see if we can figure it out.

```
Simplify[f'''''[0]]/5!
Simplify[f''''''[0]]/6!
Simplify[f'''''''[0]]/7!
Simplify[f''''''''[0]]/8!
```

We find $c_5 = 0$, $c_6 = 1/2$, $c_7 = 0$, and $c_8 = -1/6$. We now have

$$f(x) = x^2 - x^4 + \frac{1}{2}x^6 - \frac{1}{6}x^8 + \cdots$$

Our pattern is

$$f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!} x^{2n-2}.$$

Let's get the radius of convergence using the ratio test: where $a_n = \dfrac{(-1)^n}{(n-2)!} x^{2n-2}$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n}}{(n-1)!} \cdot \frac{(n-2)!}{(-1)^n x^{2n-2}} \right| \\
&= \lim_{n\to\infty} \left| x^{2n+2-2n} \cdot \frac{1}{n-1} \right| \\
&= |x^2| \lim_{n\to\infty} \frac{1}{n-1} \\
&= |x^2| \cdot 0 = 0 < 1 \quad \text{for all } x.
\end{aligned}$$

The radius of convergence is $R = \infty$. We have shown that

$$f(x) = x^2 e^{-x^2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!} x^{2n-2}, \quad |x| < \infty.$$

Now we can do the integral, which will work since the integration limits are inside $|x| < \infty$:

$$\begin{aligned}
\int_0^{1/2} x^2 e^{-x^2}\, dx &= \int_0^{1/2} \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!} x^{2n-2}\, dx \\
&= \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!} \int_0^{1/2} x^{2n-2}\, dx \\
&= \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!} \left. \frac{x^{2n-1}}{2n-1} \right|_0^{1/2} \\
&= \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!} \left(\frac{(1/2)^{2n-1}}{2n-1} - \frac{0^{2n-1}}{2n-1} \right) \\
&= \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!(2n-1)2^{2n-1}}
\end{aligned}$$

Since the series is alternating, we can estimate the error in truncating by the first term dropped, $|R_n| \le |a_{n+1}|$. Here is some more *Mathematica* help:

```
a[n_] = (-1)^n/(n - 2)!/(2n - 1)/2.0^(2n - 1)
a[2]
a[3]
a[4]
a[5]
a[2] + a[3] + a[4]
```

$$
\begin{aligned}
\int_0^{1/2} x^2 e^{-x^2}\,dx &= \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!(2n-1)2^{2n-1}} \\
&\sim 0.0416667 - 0.00625 + 0.000558036, \quad |\text{error}| < 0.000036169. \\
&\sim 0.03597, \quad |\text{error}| < 0.00004.
\end{aligned}
$$

There are other ways to go about solving this problem. This was the direct, brute force method of getting a Taylor series for the integrand.

We might also have used the Taylor series for e^y, and modified it to get the series for e^{-x^2}, and then multiplied that by x^2 to get the Taylor series for $x^2 e^{-x^2}$.