

Quadratic Functions

Obtaining vertex form by completing the square

Using the algebraic technique of completing the square, we can convert easily from the form $f(x) = ax^2 + bx + c$ to the vertex form $f(x) = a(x - h)^2 + k$.

$$\begin{aligned}
 f(x) &= ax^2 + bx + c \\
 &= a \left(x^2 + \frac{b}{a}x \right) + c \\
 &= a \left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a} \right)^2 - \left(\frac{b}{2a} \right)^2 \right) + c && \text{the blue terms add to zero; we haven't changed the equality} \\
 &= a \left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a} \right)^2 - \left(\frac{b}{2a} \right)^2 \right) + c && \text{the red terms can be collected together} \\
 &= a \left(\left[x + \frac{b}{2a} \right]^2 - \left(\frac{b}{2a} \right)^2 \right) + c \\
 &= a \left[x + \frac{b}{2a} \right]^2 - a \left(\frac{b}{2a} \right)^2 + c && \text{simplify} \\
 &= a \left[x + \frac{b}{2a} \right]^2 - \frac{b^2}{4a} + c \\
 &= a \left[x + \frac{b}{2a} \right]^2 - \frac{b^2 - 4ac}{4a} \\
 &= a \left[x - \left(-\frac{b}{2a} \right) \right]^2 - \frac{b^2 - 4ac}{4a}
 \end{aligned}$$

This is now in the vertex form, and we can identify the vertex as $(h, k) = \left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a} \right)$

and the axis of symmetry as $x = -\frac{b}{2a}$.

Do not memorize these formulas, instead understand the procedure, and work it through each time.

Example Find the vertex of $f(x) = 14x^2 - 3x + 47$.

$$\begin{aligned}
 f(x) &= 14x^2 - 3x + 47 \\
 &= 14 \left(x^2 + \left(\frac{-3}{14} \right) x \right) + 47 \\
 &= 14 \left(x^2 + \left(\frac{-3}{14} \right) x + \left(\frac{-3}{28} \right)^2 - \left(\frac{-3}{28} \right)^2 \right) + 47 \\
 &= 14 \left(x^2 + \left(\frac{-3}{14} \right) x + \left(\frac{-3}{28} \right)^2 - \left(\frac{-3}{28} \right)^2 \right) + 47 \\
 &= 14 \left(\left[x + \left(\frac{-3}{28} \right) \right]^2 - \left(\frac{-3}{28} \right)^2 \right) + 47
 \end{aligned}$$

$$\begin{aligned}
&= 14 \left(\left[x - \frac{3}{28} \right]^2 - \left(\frac{-3}{28} \right)^2 \right) + 47 && \text{notice how our treatment of the minus sign for } b \text{ was done!} \\
&= 14 \left[x - \frac{3}{28} \right]^2 - 14 \left(\frac{-3}{28} \right)^2 + 47 && \text{simplify} \\
&= 14 \left[x - \frac{3}{28} \right]^2 - \frac{126}{784} + 47 \\
&= 14 \left[x - \frac{3}{28} \right]^2 - \frac{9}{56} + \frac{2632}{56} \\
&= 14 \left[x - \frac{3}{28} \right]^2 - \frac{2623}{56}
\end{aligned}$$

This is now in the vertex form, and we can identify the vertex as $(h, k) = \left(-\frac{3}{28}, -\frac{2623}{56} \right)$ and axis of symmetry $x = -\frac{3}{28}$.

Obtaining the x -intercepts from the vertex form

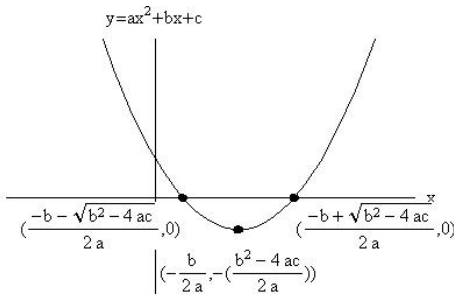
Once we have the vertex form, we can solve $f(x) = 0$ and obtain the x -intercepts.

$$\begin{aligned}
f(x) = 0 &= a \left[x - \left(-\frac{b}{2a} \right) \right]^2 - \frac{b^2 - 4ac}{4a} \\
0 &= a \left[x + \frac{b}{2a} \right]^2 - \frac{b^2 - 4ac}{4a} \\
a \left[x + \frac{b}{2a} \right]^2 &= \frac{b^2 - 4ac}{4a} \\
\left[x + \frac{b}{2a} \right]^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x + \frac{b}{2a} &= \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

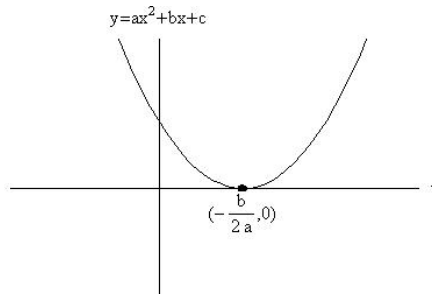
Now we understand where the quadratic formula comes from.

Sketches of quadratic functions

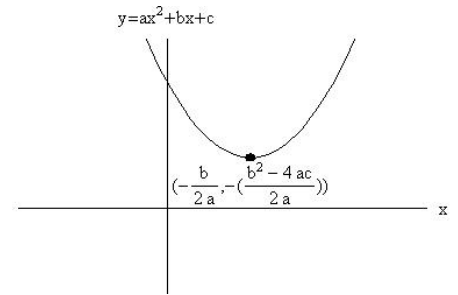
Knowing the sign of $b^2 - 4ac$ can allow us to draw accurate sketches of quadratic functions $f(x) = ax^2 + bx + c$. In what follows we assume $a > 0$ so we know the quadratic function is opening up.



$b^2 - 4ac > 0$
two real valued x -intercepts



$b^2 - 4ac = 0$
one real valued x -intercept
(of multiplicity 2)



$b^2 - 4ac < 0$
no real valued x -intercepts

Test Your Understanding

Problem: Show $f(x) = 13x^2 - x - 1 = 13\left(x - \frac{1}{26}\right)^2 - \frac{53}{52}$. Identify the vertex and the x -intercepts.

Problem: Show $f(x) = x^2 - \frac{1}{2}x - \frac{1}{9} = \left(x - \frac{1}{4}\right)^2 - \frac{25}{144}$. Identify the vertex and the x -intercepts, then use this information to sketch the graph without using a calculator. Verify your result using a calculator.

Problem: Show $f(x) = 5x^2 + 6x + 7 = 5\left(x + \frac{3}{5}\right)^2 + \frac{26}{5}$. Using the vertex form, explain how you know there are no x -intercepts for this function.

Further Resources

There is an excellent webpage that can help you practice completing the square:

<http://www.csm.astate.edu/algebra/complete.html>

(this should be an active link, click on it to go to the site)