Name: Answer Key

Instructions:

- Attempt all questions.
- The test is out of 100 marks.
- There are 5 questions, 20 marks each.
- You have 65 minutes to complete the test.

Advice:

- Budget your time.
- First, do those questions which you know how to do immediately.
- Leave questions which you find difficult until last.
- Ask for clarification if you do not understand a question.
- You must show your work. Label sketches well (points of interest, asymptotes, axes, etc).

Good Luck!

Problem 1. (20 marks) Solve $\cos 2x + \cos x = 0$ algebraically for exact solutions in the interval $[0, 2\pi)$.

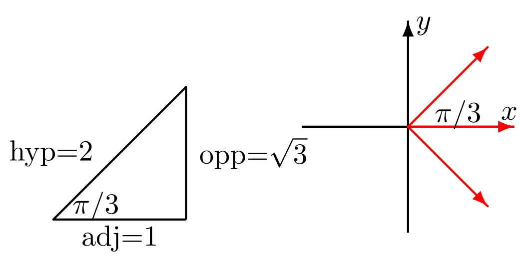
Solution:

$$\begin{aligned} \cos 2x + \cos x &= \cos^2 x - \sin^2 x + \cos x \\ &= \cos^2 x - (1 - \cos^2 x) + \cos x \\ &= 2\cos^2 x + \cos x - 1 = 0 \end{aligned}$$

Let $y = \cos x$. Then

$$\begin{aligned} \cos 2x + \cos x &= 2\cos^2 x + \cos x - 1 = 0 \\ &= 2y^2 + y - 1 = 0 \\ y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = \frac{2}{4} \text{ or } \frac{-4}{4} = \frac{1}{2} \text{ or } -1 \end{aligned}$$

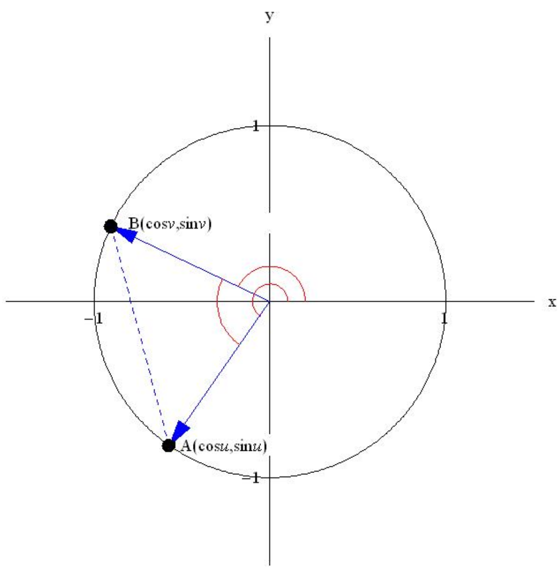
So we must solve $y = \cos x = 1/2$ and $y = \cos x = -1$.
The equation $\cos x = -1$ has a solution of π in the interval $[0, 2\pi)$.
The equation $\cos x = \text{adj}/\text{hyp} = 1/2$ corresponds to one of our special triangles:

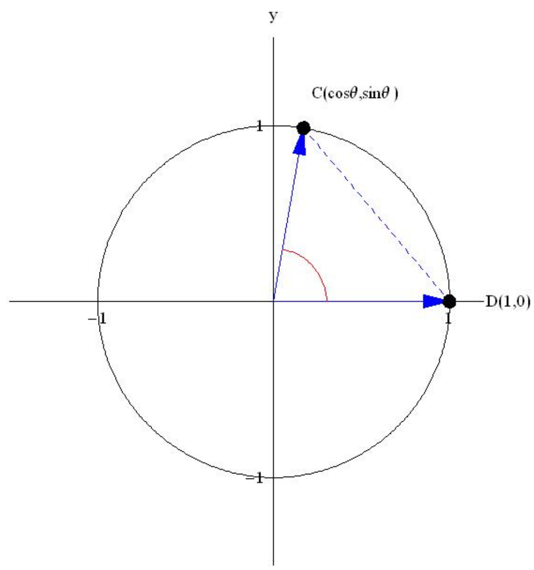


So the solution is $\pi/3$.
There is also a solution in Quadrant IV at $2\pi - \pi/3 = 5\pi/3$ in the interval $[0, 2\pi)$.
The solutions to $\cos 2x + \cos x = 0$ in the interval $[0, 2\pi)$ are $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

Problem 2. (20 marks) Prove the trig identity $\cos(u-v) = \cos u \cos v + \sin u \sin v$ using the following process:
Draw a diagram of the unit circle which includes the angles u, v, and $\theta = u - v$.
Draw a second diagram with the angle θ in standard position.
Use the distance formula and simplify to get the given identity.

Solution: The angle u leads to a point $A(\cos u, \sin u)$ on the unit circle.
The angle v leads to a point $B(\cos v, \sin v)$ on the unit circle.
The angle $\theta = u - v$ is the angle between the the terminal sides of u and v.
The dotted line connects the points A and B.





We can rotate the geometry of this picture so that the angle θ is in standard position.
The dashed lines are the same length in both pictures. Therefore, we can use the distance between two points formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, and we can write:

$$\sqrt{(\cos u - \cos v)^2 + (\sin u - \sin v)^2} = \sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2}$$

Remember, $\theta = u - v$, so we want to solve this for $\cos\theta = \cos(u - v)$.

$$\begin{aligned}
(\sqrt{(\cos u - \cos v)^2 + (\sin u - \sin v)^2})^2 &= (\sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2})^2 \\
(\cos u - \cos v)^2 + (\sin u - \sin v)^2 &= (\cos\theta - 1)^2 + (\sin\theta)^2 \\
(\cos^2 u + \cos^2 v - 2\cos u \cos v) + (\sin^2 u + \sin^2 v - 2\sin u \sin v) &= (\cos^2\theta + 1 - 2\cos\theta) + \sin^2\theta \\
(\cos^2 u + \sin^2 u) - 2\cos u \cos v + (\cos^2 v + \sin^2 v) - 2\sin u \sin v &= (\cos^2\theta + \sin^2\theta) + 1 - 2\cos\theta \\
(1) - 2\cos u \cos v + (1) - 2\sin u \sin v &= (1) + 1 - 2\cos\theta \\
2 - 2\cos u \cos v - 2\sin u \sin v &= 2 - 2\cos\theta \\
-2\cos u \cos v - 2\sin u \sin v &= -2\cos\theta \\
+\cos u \cos v + \sin u \sin v &= +\cos\theta \\
\cos\theta = \cos(u - v) &= \cos u \cos v + \sin u \sin v
\end{aligned}$$

We have arrived at the trig identity $\boxed{\cos(u-v) = \cos u \cos v + \sin u \sin v}$.

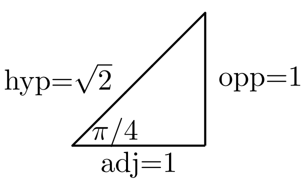
Problem 3. (20 marks) Find the value of $\sin\left(-\frac{\pi}{12}\right)$ exactly using an angle difference formula.

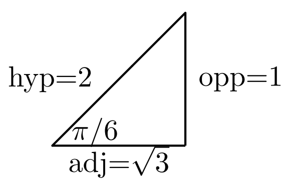
Solution:
First, we need to figure out how to relate $-\pi/12$ to some of our special angles, since we are told to find this answer exactly.

$$\frac{-\pi}{12} = \frac{-2\pi}{24} = \frac{4\pi - 6\pi}{24} = \frac{\pi}{6} - \frac{\pi}{4}.$$

Therefore,

$$\begin{aligned}
\sin\left(-\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{6} - \frac{\pi}{4}\right) \\
&= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right), \quad \text{use } \sin(u-v) = \sin u \cos v - \cos u \sin v \\
&= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right), \quad \text{using reference triangles below} \\
&= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}
\end{aligned}$$



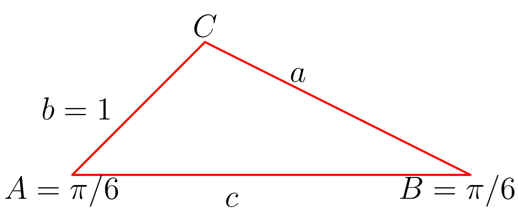


Problem 4. (20 marks) Use the power reducing identities to prove the identity $\sin^4 x = \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$.

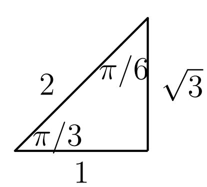
Solution:

$$
\begin{aligned}
\sin^4 x &= \left(\sin^2 x\right)^2 \\
&= \left(\frac{1-\cos 2x}{2}\right)^2, \quad \text{using } \sin^2 u = \frac{1-\cos 2u}{2}, \text{ with } u = x. \\
&= \frac{1}{4}\left(1-\cos 2x\right)^2 \\
&= \frac{1}{4}\left(1+\cos^2 2x - 2\cos 2x\right) \\
&= \frac{1}{4}\left(1+\left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right), \quad \text{using } \cos^2 u = \frac{1+\cos 2u}{2}, \text{ with } u = 2x. \\
&= \frac{1}{4}\left(\frac{2}{2}+\frac{1+\cos 4x}{2} - \frac{4\cos 2x}{2}\right) \\
&= \frac{1}{8}\left(2+1+\cos 4x - 4\cos 2x\right) \\
&= \frac{1}{8}\left(3+\cos 4x - 4\cos 2x\right) \\
&= \frac{1}{8}\left(3-4\cos 2x + \cos 4x\right)
\end{aligned}
$$

Problem 5. (20 marks) Solve the following triangle (determine a, c and C exactly). Note the triangle is not drawn to scale.



Reference Triangle:



Solution:
When you know two angles, it is easy to get the third: $C = \pi - \frac{\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3}$.
Now we can determine a using the Law of Sines:

$$
\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin B}{b} \\
a &= b\frac{\sin A}{\sin B} = (1)\frac{\sin \pi/6}{\sin \pi/6} = 1
\end{aligned}
$$

We can determine c. Note: $\sin(2\pi/3) = 2\sin(\pi/3)\cos(\pi/3) = 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{\sqrt{3}}{2}$, using reference triangle above.

$$
\begin{aligned}
\frac{\sin C}{c} &= \frac{\sin B}{b} \\
c &= b\frac{\sin C}{\sin B} = (1)\frac{\sin 2\pi/3}{\sin \pi/6} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = \sqrt{3}
\end{aligned}
$$