Example 3.5.33 Solve the equation $\dfrac{500}{1+25e^{0.3x}} = 200$ algebraically.

$$\begin{aligned}
\frac{500}{1+25e^{0.3x}} &= 200 \quad \text{(isolate the } e^{0.3x}\text{)} \\
\frac{1+25e^{0.3x}}{500} &= \frac{1}{200} \\
1+25e^{0.3x} &= \frac{500}{200} \\
25e^{0.3x} &= \frac{5}{2} - 1 \\
25e^{0.3x} &= \frac{3}{2} \\
e^{0.3x} &= \frac{3}{50} \quad \text{(take the natural logarithm of both sides)} \\
\ln e^{0.3x} &= \ln\left(\frac{3}{50}\right) \quad \text{(use the logarithm/exponential inverse function property)} \\
0.3x &= \ln\left(\frac{3}{50}\right) \\
x &= \frac{1}{0.3}\ln\left(\frac{3}{50}\right) \\
&= \frac{10}{3}\ln\left(\frac{3}{50}\right) \\
&\sim = 9.37804
\end{aligned}$$

Example 3.5.38 Solve the equation $\log(x-2) + \log(x+5) = 2\log 3$ algebraically.

$$\begin{aligned}
\log(x-2) + \log(x+5) &= 2\log 3 \\
\log((x-2)(x+5)) &= \log 3^2 \\
10^{\log(x-2)(x+5)} &= 10^{\log 9} \\
(x-2)(x+5) &= 9 \\
x^2 + 3x - 10 &= 9 \\
x^2 + 3x - 19 &= 0
\end{aligned}$$

Use the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-19)}}{2(1)} \\
&= \frac{-3 \pm \sqrt{9+76}}{2} \\
&= \frac{-3 \pm \sqrt{85}}{2}
\end{aligned}$$

So the solution is $x = (-3 - \sqrt{85})/2 \sim -6.10977$ and $x = (-3 + \sqrt{85})/2 \sim 3.10977$. However, one of these is extraneous. From the original equation, we must have $x - 2 \geq 0$ and $x + 5 \geq 0$ for the logarithms to be defined. Remember, $\log_b x$ is defined only for $x \geq 0$. These conditions are satisfied if $x \geq 2$, and the only solution to the problem is $x = (-3 + \sqrt{85})/2 \sim 3.10977$.

Example 3.5.31 This is a bonus question–a difficult type of equation to solve! Solving this involves noticing that the equation is quadratic in e^x.

Solve the equation $\dfrac{e^x + e^{-x}}{2} = 4$ algebraically.

$$\begin{aligned}
\frac{e^x + e^{-x}}{2} &= 4 \\
e^x + e^{-x} &= 8 \\
e^x + e^{-x} - 8 &= 0 \\
e^x(e^x + e^{-x} - 8 &= 0) \\
e^{2x} + 1 - 8e^x &= 0 \\
e^{2x} - 8e^x + 1 &= 0 \\
e^{2x} - 8e^x + 1 &= 0
\end{aligned}$$

This is now a quadratic in e^x, let $z = e^x$, and we write

$$\begin{aligned}
z^2 - 8z + 1 &= 0 \\
z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{+8 \pm \sqrt{(-8)^2 - 4(1)(1)}}{2(1)} \\
&= \frac{8 \pm \sqrt{60}}{2} \\
&= 4 \pm \sqrt{15}
\end{aligned}$$

Therefore,

$$\begin{aligned}
z = e^x &= 4 \pm \sqrt{15} \\
\ln e^x &= \ln(4 \pm \sqrt{15}) \\
x &= \ln(4 \pm \sqrt{15})
\end{aligned}$$

Note that $\ln(e^x + e^{-x}) = \ln 8$ is a mathematical dead end. We can't solve for x from this.