Example 2.3.41 State the degree and list the zeros of the polynomial function. State the multiplicity of each zero and whether the graph crosses the x-axis at the corresponding x-intercept. Then sketch the graph of the polynomial function by hand.

$$f(x) = (x-1)^3(x+2)^2$$

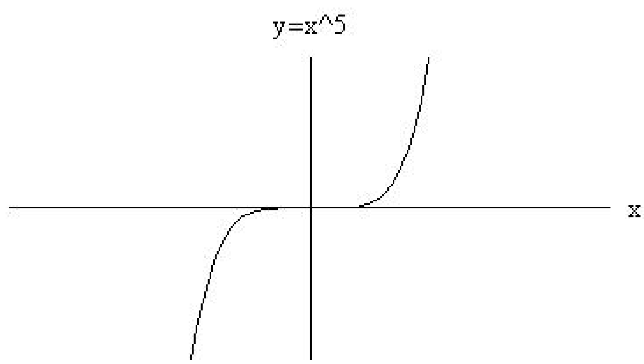
This is a fifth degree polynomial, so it will have at most 5 real valued roots and 4 local extrema.

The polynomial will have two zeros, at $x = -2, 1$.
The polynomial will change sign (cross the x axis) at the roots with odd multiplicity; these roots are
$x = 1$ multiplicity 3,
The polynomial will not change sign (cross the x axis) at the roots with even multiplicity; these roots are
$x = -2$ multiplicity 2.
The end behaviour of the polynomial is found by determining the leading term, which is

$$(x-1)^3(x+2)^2 \sim (x)^3(x)^2 = x^5 \text{ for large } |x|.$$

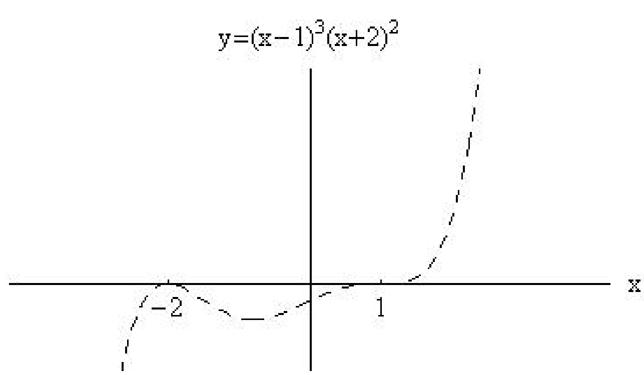
The end behaviour of the monomial x^5 is

$$\lim_{x\to-\infty} x^5 = -\infty \qquad \lim_{x\to\infty} x^5 = \infty$$



This sketch tells us the end behaviour of the polynomial.

Putting it all together, we can sketch the polynomial f



Example 2.3.56 Using only algebra, find a cubic function with the given zeros.

$$1, 1+\sqrt{2}, 1-\sqrt{2}$$

We can get the cubic from the zeros by writing the factored form, where c_i are the roots. In this case, we are given all three roots, so each root is multiplicity one.

$$\begin{aligned}
f(x) &= (x-c_1)(x-c_2)(x-c_3) \\
&= (x-1)(x-(1+\sqrt{2}))(x-(1-\sqrt{2})) \\
&= (x-1)(x^2-2x-1) \\
&= (x^3-2x^2-x)-(x^2-2x-1) \\
&= x^3-2x^2-x-x^2+2x+1 \\
&= x^3-3x^2+x+1
\end{aligned}$$

I have simplified to standard form for a polynomial.

Check our answer by substituting in the roots:

$$\begin{aligned}
f(1) &= (1)^3-3(1)^2+(1)+1 \\
&= 0 \\
f(1+\sqrt{2}) &= (1+\sqrt{2})^3-3(1+\sqrt{2})^2+(1+\sqrt{2})+1 \\
&= (1+3\sqrt{2}+3(\sqrt{2})^2+(\sqrt{2})^3)-3(1+2\sqrt{2}+(\sqrt{2})^2)+(1+\sqrt{2})+1 \\
&= 1+3\sqrt{2}+6+2\sqrt{2}-3-6\sqrt{2}-6+1+\sqrt{2}+1 \\
&= 0 \\
f(1-\sqrt{2}) &= (1-\sqrt{2})^3-3(1-\sqrt{2})^2+(1-\sqrt{2})+1 \\
&= (1-3\sqrt{2}+3(\sqrt{2})^2-(\sqrt{2})^3)-3(1-2\sqrt{2}+(\sqrt{2})^2)+(1-\sqrt{2})+1 \\
&= 1-3\sqrt{2}+6-2\sqrt{2}-3+6\sqrt{2}-6+1-\sqrt{2}+1 \\
&= 0
\end{aligned}$$