4452 Mathematical Modeling Lecture 15: Deterministic and Stochastic Battle Simulations

Let's examine a battle simulation. Say that we have an two forces, one of 3000 soldiers (Red), and one of 3000 soldiers (Blue). The Blue army has superior weapons than the Red army, which results in a Blue soldier being 1.5 times more effective than a Red soldier. However, the Blue army's weapons suffer increased malfunctions under severe weather conditions, while the red army's weapons are essentially unaffected by weather. The result is that under severe weather conditions, a soldier in the Red army is more effective than a Blue soldier. Model this situation. [1]

Deterministic Model

More effective, in this case, means able to kill more opposing soldiers. We will assume that the battle is largely carried out by direct fire contact. The number of troops alive at any time will be the number of troops alive at the past time minus the number of deaths. Since we are dealing with individual soldiers, we must use integer values for the number of soldiers.

We can use a time step of 0.1 hours and model this as a discrete dynamical system. In good weather, our assumptions mean that in one hour each surviving Red soldier kills 0.1 Blue soldier, and each surviving Red soldier kills 0.15 Red soldiers. Therefore, in one time step, Each Red soldier kills 0.01 Blue soldiers, and each Blue soldier kills 0.015 Red soldier. Of course, these numbers are ones that I picked, but their ratio is the important bit. Any practical application would have more data to help you choose these numbers. This translates into the equations:

$$
\begin{aligned}
x_{n+1} &= x_n - 0.015 y_n \\
y_{n+1} &= y_n - 0.010 x_n,
\end{aligned}
$$

where x is the number of Red Soldiers and y is the number of Blue soldiers.

We can initialize this model with $x_0 = 3000$, and $y_0 = 3000$. At each time step, we must ensure that x and y are integers. This is simple to solve (see the *Mathematica* file), and we find that the battle takes 9.4 hours to complete, with Blue winning with 1716 soldiers surviving. A graph of the battle is presented in Fig. 1.

This method is deterministic, in the sense that once we set our parameters we get the same solution every time.

Adding Weather: A Stochastic Effect

Now, how about adding weather to this problem? Since the efficiency of the Blue soldiers decreases in bad weather, but the efficiency of the Red soldiers remains unchanged, we can model this by introducing the parameter w:

$$
\begin{aligned}
x_{n+1} &= x_n - w_n 0.015 y_n \\
y_{n+1} &= y_n - 0.010 x_n, \\
w_{n+1} &= w_n + 0.1 * R
\end{aligned}
\tag{1}
$$

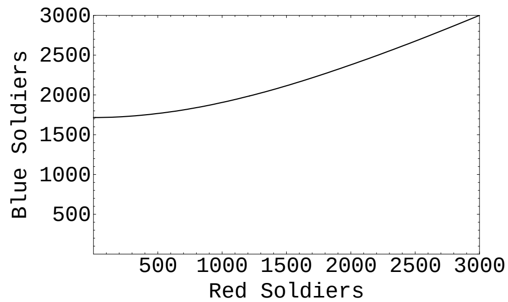

Figure 1: The result of running the deterministic battle simulation which assumes weather remains unchanged.

where w_n is between 0 and 1, and $w_n = 1$ corresponds to good weather, and $w_n = 0$ corresponds to poor weather. At each time step R is randomly chosen to be $R = -1$ (weather gets worse), $R = 0$ (weather stays the same), or $R = 1$ (weather improves). Notice that this is an extreme sensitivity to weather, for in poor weather the effectiveness of the Blue forces is seriously reduced.

A long run of poor weather could be enough to turn the battle in Red's favour. However, the poor weather must occur during the first portion of the battle, or Red's forces will be sufficiently decimated that even a long run of poor weather later on will not be enough to allow them to win the battle. The Blue commander should feel confident of beginning the battle on a sunny day, and allowing their weapon superiority during good weather to carry them to victory.

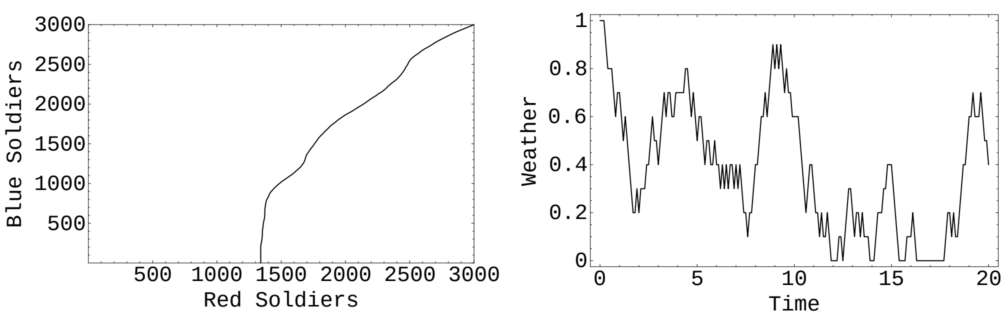

Figure 2: The result of one run from the battle simulation which includes changing weather conditions (Eq. (1)). The poor weather at the beginning of the battle allowed the Red troops to win the battle.

This is now a simulation model, where we need to do a great many runs to determine the outcome of the battle under many different weather scenarios. There is no other way to do this than to carry out many simulations and look for emerging patterns. The fact that poor weather at the start of the battle is necessary to help Red win was noticed after many trial runs. In 20 simulations, Red won 8, Blue won 7, and 5 battles had not ended after 20 hours.

Looking at the weather plot in Fig. 2, we may question our model of the weather. In less less than one day the weather fluctuates dramatically, which is probably not a reasonable thing to happen. The weather should not go from $w = 1$ to $w = 0$ and then back to $w = 1$ in a single day. We should make the weather change less dramatically at each time step,

$$x_{n+1} = x_n - w_n 0.015 y_n$$

$$
\begin{aligned}
y_{n+1} &= y_n - 0.010 x_n, \\
w_{n+1} &= w_n + 0.03 * R. \qquad (2)
\end{aligned}
$$

Also, previously we assumed that we began with good weather, $w = 1$. It may be more realistic to assume that the weather is not perfect (whatever that means) at the start of the battle. So we will initialize by saying $w = 0.8$. The results from a single run of this situation are contained in Fig. 3. In twenty simulations of this type, all the battles ended after 30 hours, Red won 3 and Blue won 17.

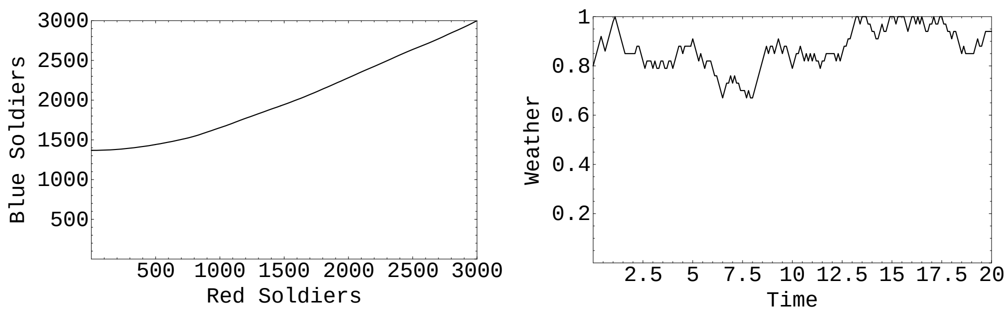

Figure 3: The result of one run from the battle simulation which includes changing weather conditions (Eq. (2)).

These considerations of how weather behaves probably produce a more realistic model of the battle. However, if the climate of the battle field is known, the weather characteristics could be tailored to the region. Since weather plays an important role in the outcome of the battle, this is an important analysis that you should make.

A Stochastic Model

Instead of deterministic, we could model this situation as a stochastic model. At any stage, the next casualty in the battle should come from either the Red side or the Blue side, that is, the battle should proceed by individual kills, not a percentage kill as we have modeled above. How could we construct a model that incorporates this? For the moment, let's go back to assuming that the weather is always good, which we treated in a deterministic manner earlier.

At any time during the battle, the probability that the next death is from the Red squad is

$$\frac{0.15y}{0.15y + 0.1x},$$

and the probability that a Blue soldier dies next is

$$\frac{0.1x}{0.15y + 0.1x}.$$

The probabilities add up to 1. The ratio of the probability of deaths is:

$$\frac{\left(\frac{0.15y}{0.15y+0.1x}\right)}{\left(\frac{0.1x}{0.15y+0.1x}\right)} = 1.5\frac{y}{x}$$

which means that the Red (y) will be killed at a rate 1.5 times that of the Blue (x).

Here is how we proceed to solve. We begin with our initial soldier populations. We calculate the probability that a Red soldier will be killed next, which is

$$P(x, y) = \frac{0.15y}{0.15y + 0.1x}$$

$$P(3000, 3000) = 0.6$$

We then get a random number in $(0, 1)$. If the random number is less than 0.6, then a Red soldier is the next killed, and we write:

$$\begin{aligned} x_{n+1} &= x_n - 1 \\ y_{n+1} &= y_n, \end{aligned}$$

otherwise a Blue soldier is killed next, and we write

$$\begin{aligned} x_{n+1} &= x_n \\ y_{n+1} &= y_n - 1. \end{aligned}$$

Our new populations are used to calculate new probabilities, and the procedure is repeated. This goes on until either the Red or Blue squad is reduced to zero.

This is also a simulation, since different runs will produce different outcomes. The result from a single run is reported in Fig. 4, which looks very similar to the deterministic model we used earlier, which is shown in Fig. 1.

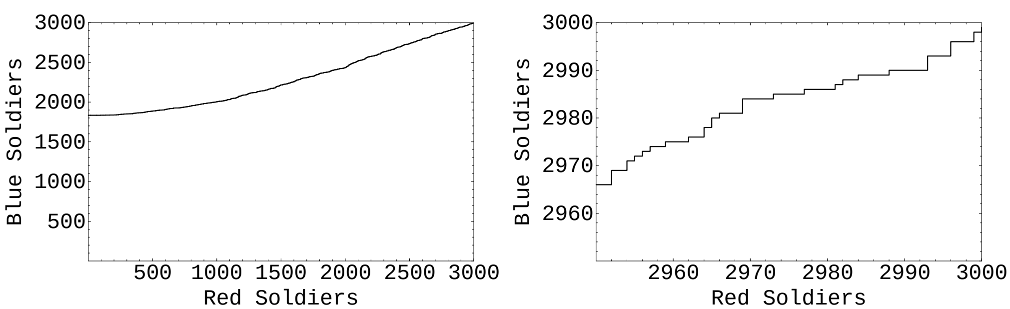

Figure 4: The result of running the stochastic battle simulation which assumes weather remains unchanged. The graph on the right is a close up, showing the step wise nature of the graph. Since there are random numbers in the stochastic simulation, it will be different for each run.

I performed 100 runs of the simulation, and determined the final number of Blue soldiers left after each battle. The results are contained in the histograms in Fig. 5. The fact that our final distribution appears to be a normal distribution can be explained by the central limit theorem. If we took the number of runs $n \to \infty$ we would achieve a normal distribution of the number of Blue soldiers left after the battle. The mean we find is 1725 Blue soldiers remaining, which compares well with the figure of 1716 Blue soldiers remaining we found in the deterministic model. I did the fitting to a normal distribution simply by eye, and if it where important enough you could perform more significant statistical analysis on the data that was collected from the 100 runs.

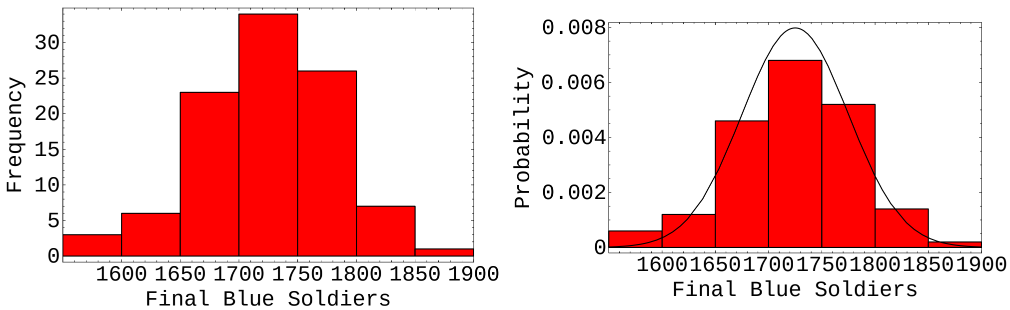

Figure 5: Histograms showing the result of running the stochastic battle simulation which assumes weather remains unchanged 100 times. The graph on the left is overlayed with a normal distribution with mean $\mu = 1725$ and $\sigma = 50$. It appears that we have a mean of about 1725 Blue soldiers left after the battle.

The addition of weather to the stochastic model

There is a problem with adding weather to the stochastic model we have just created. Nowhere in the model does time appear. In fact, we have no way of knowing how long these battles take, unlike the deterministic model for which time intrinsically involved. If we need to vary the probabilities based on weather, we first need to add time to our stochastic model.

In the deterministic model, we made the assumption that the kill rate was per hour. We haven't yet used that information in the stochastic model. We can introduce time by assuming that the deaths occurred in a Poisson process fashion,

$$Pr(\text{ death in time interval } \Delta t\ ) = e^{-\Delta t/\mu}.$$

This would mean that at any time during the battle the time between deaths is a negative exponential distributed random variable with mean $\mu = 1/(0.15y + 0.10x)$. That is, the more people we have, the quicker deaths occur; the smaller the number of combatants, the longer the time between deaths. So at the beginning of the battle deaths would occur more frequently than at the end of the battle. The time between deaths is

$$\Delta t = -\frac{\ln R}{0.15y + 0.10x}$$

where R is a random number between 0 and 1.

If we do this, we find for our first run that the battle ended after 8.87 hours. This agrees fairly well with the result from the deterministic model, which had the battle ending in 9.4 hours. "Agrees well" here means we are on the same time scale, and the battle ends in under a day. Our stochastic model will have different times for each run, so again we should rerun a number of times and see what the distribution is for the duration of the battle. The results of 100 simulations are contained in Fig. 6.

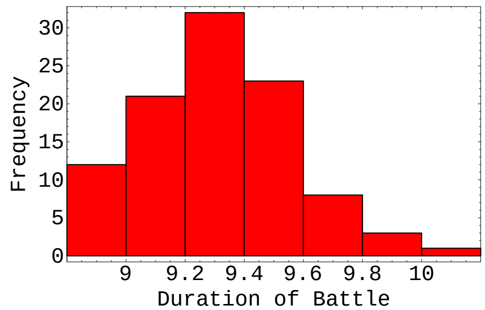

Figure 6: Histogram showing the result of running the stochastic battle simulation which assumes weather remains unchanged 100 times. It appears that we have a mean of 9.3 hours for the battle, which agrees well with the deterministic model estimate of 9.4 hours.

Here is how we proceed to solve when we want to incorporate weather into our model. We begin with our initial soldier populations, $x = 3000$, $y = 3000$, and the weather begins at $w = 0.8$. We calculate the probability that a Red soldier will be killed next, which is

$$P(x, y) = \frac{w0.15y}{w0.15y + 0.1x}$$

We then get a random number in $(0, 1)$. If the random number is less than the above probability, then a Red soldier is the next killed, and we write:

$$\begin{aligned} x_{n+1} &= x_n - 1 \\ y_{n+1} &= y_n, \end{aligned}$$

otherwise a Blue soldier is killed next, and we write

$$\begin{aligned} x_{n+1} &= x_n \\ y_{n+1} &= y_n - 1. \end{aligned}$$

The time which this death occurred at is given by

$$\Delta t = -\frac{\ln R}{w0.15y + 0.10x}$$

The weather changes according to the relation

$$w = w + 0.03 * R$$

every 0.1 hours to be consistent with our previous weather analysis. At each time step R is randomly chosen to be $R = -1$ (weather gets worse), $R = 0$ (weather stays the same), or $R = 1$ (weather improves). We could determine how many 0.1 time steps have passed and increment the weather that many times, or we could just leave the weather change as it is–it won't be quite the same as what we used before, but since these are stochastic models that will not be a problem (each separate run in a stochastic model is different anyway). We should still check our modeling of the weather, and decide if it needs to be modified.

Our new populations are used to calculate new probabilities, and the procedure is repeated. This goes on until either the Red or Blue squad is reduced to zero.

When I did a run using the above formalism, I noticed that now that we are stepping between individual kills rather than time steps of 0.1 hours, my model of the weather was atrocious. The weather was changing much too rapidly to be realistic. I therefore changed my weather to be modeled by

$$w = w + 0.005 * R$$

This produced weather fluctuations over a 24 hour period which seem reasonable to me. I performed 20 simulations of the battle. As we found before, Red could win the battle if the weather conditions were bad. All the battles ended in 25 hours, except one battle which Red won which took 36 hours to complete, and the weather during the initial stages of the battle were important if Red was to win. The weather had to get bad quickly, or the Red forces would not be able to overcome the good weather weapon superiority of the Blue forces. In these simulation, Red won 4 battles, and Blue won 16.

By changing how the weather was modeled, it is not possible to directly compare the deterministic model we first produced and the stochastic model when weather is involved. However, both simulations produced the same conclusions. The Red forced need bad weather to win, and they need bad weather at the beginning of the battle to overcome the Blue forces superiority in good weather. Beginning with 3000 soldiers on each side, Blue will win 80–85% of the battles in under 24 hours. Some results from the simulation are contained in Fig. 7.

The amount of statistical analysis you wish to spend on the problem is up to you. You could also work out mean battle times as we did before.

References

[1] D. Edwards & M. Hamson, Guide to Mathematical Modelling, CRC Press (Baco Raton) 1990 (p249).

[2] W. Mendenhall, Introduction to Probability and Statistics, Prindle, Weber, & Schmidt (Boston) 1983.

[3] D. D. Mooney & R. J. Swift, A Course In Mathematical Modeling, The Mathematical Association of America, 1999.

[4] M. Meerschaert, Mathematical Modelling, 2nd ed., Academic Press (San Diego) 1999.

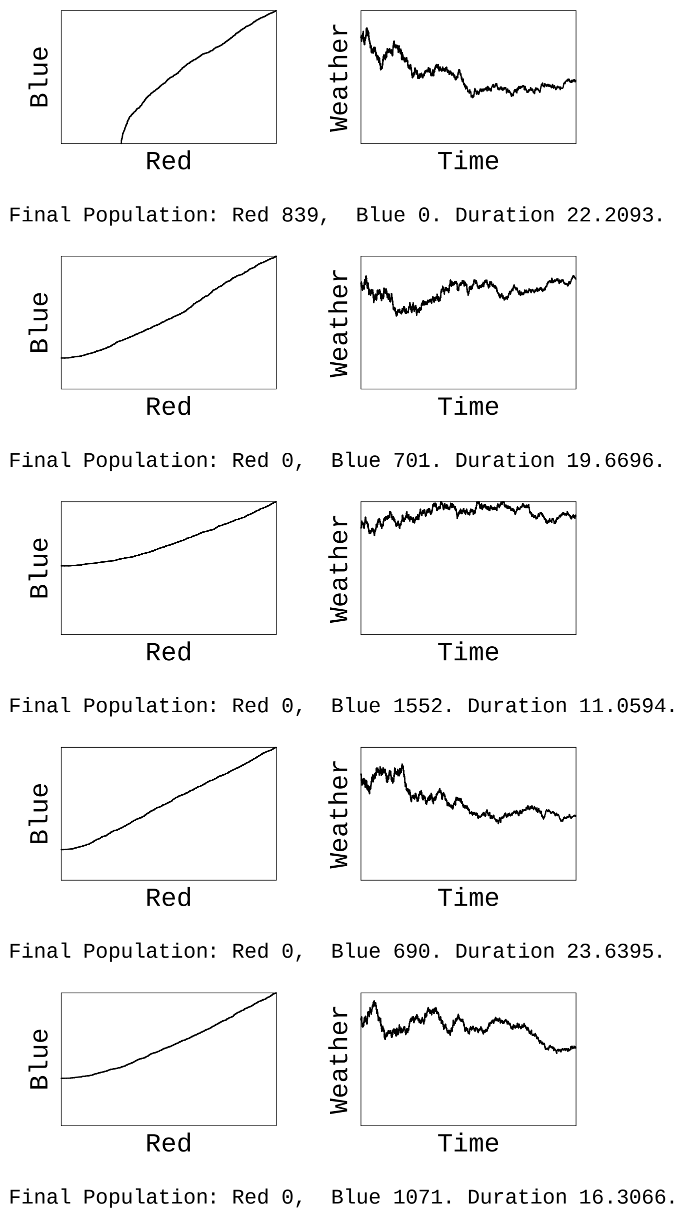

Figure 7: Five results from the stochastic battle simulation including weather.