Chapter 4: Understanding Interest Rates

Measuring Interest Rates

Four kinds of credit market instruments

**Simple loans**
principal and interest returned at maturity

**Fixed-payment loans**
e.g. mortgages, car loans

**Coupon bond**
periodic coupon payment + face value payout at maturity

Note: "coupon rate" = (coupon payment)/(face value)

**Discount bond** ("zero-coupon bond")
pays face value at maturity, sells at discount
Present Value

$x$ tomorrow is not as valuable as $x$ today

So how do we calculate the "present value" (i.e. value in today's $) of a cash flow into the future?

Example: what is a payment of $1000 after 3 years worth, in today's $?

Well, ... how much would you have to invest today at current interest rates, in order to accumulate $1000 in 3 years? That's the answer.

\[ x(1+i) = \text{amount you'd have after year 1} \]
\[ i = \text{decimal interest rate on simple loan (e.g. savings deposit)} \]

\[ [x(1+i)](1+i) = \text{amount you'd have after 2 years} \]
\[ = x[(1+i)^2] \]

\[ [(x(1+i))(1+i)](1+i) = \text{amount you'd have after 3 years} \]
\[ = x[(1+i)^3] \]

\[ = x[(1+i)^n] = \text{amount you'd have after } n \text{ years, from investing }$x \text{ @ } i \]

Now solve the following for \( x \):

\[ x[(1+i)^3] = 1000 \]
\[ x^* = (1000)/ [(1+i)^3] \]

e.g. let \( i = .1 \) (10% interest)
\[ \Rightarrow x^* = $751.31 \]

Intuitively: if what you want is $1000 in three years time, you can "manufacture" it yourself for $751.31.

\[ \Rightarrow \text{you would never pay more than } $751.31 \text{ for the right to that cash flow.} \]

We say "the present discounted value of $1000 after 3 years is $751.31"

PDV \((Y \text{ after n years}) = (Y)/[(1+i)^n] \) (assuming constant \( i \))
Yield to Maturity

Definition: the interest rate that equates the present value of payments received from a debt instrument, with its value today.

Simple Loan

let (interest on simple loan)=.1 (10%), Y=1000, n=1
⇒ after 1 year, receive payment of $1100
What interest rate makes the present value of the payment equal the current value (1000)? Solve for i:

\[
\frac{1100}{1+i}=1000
\]
\[i^*=0.1 \text{ (10%)}\]

⇒for simple loans, the yield to maturity and the simple interest rate are one and the same thing.

Fixed Payment Loan

What is the yield to maturity on a $1000 loan, paid back in 3 annual installments (after 1, 2, & 3 years) of $500?

PDV of first payment=\((500)/(1+i)\)
PDV of 2nd payment=\((500)/[(1+i)^2]\)
PDV of 3rd payment=\((500)/[(1+i)^3]\)

PDV of payment stream = \((500)/(1+i)+(500)/[(1+i)^2]+(500)/[(1+i)^3]\)

Loan value=1000

Therefore, solve for i (yield to maturity)
1000=\((500)/(1+i)+(500)/[(1+i)^2]+(500)/[(1+i)^3]\)

\[i^*=0.237 \text{ (23.7\% YTM)}\]

in general, solve

\[LV=(FP)[(1+i)^{-1}+(1+i)^{-2}+(1+i)^{-3}+...+(1+i)^{-n}]\]

where \(LV\equiv\text{loan value, } FP\equiv\text{fixed annual payment, } n\equiv\text{maturity}\)
Coupon Bond

What is the yield to maturity on a $1000 face value coupon bond, paid back in 2 annual installments (after years 1 & 2) of $300, and a liquidation payment of $1000 after the third year

\[ \text{PDV of first payment} = \frac{300}{(1+i)} \]
\[ \text{PDV of 2nd payment} = \frac{300}{(1+i)^2} \]
\[ \text{PDV of 3rd payment} = \frac{1000}{(1+i)^3} \]
\[ \text{PDV of payment stream} = \frac{300}{(1+i)} + \frac{300}{(1+i)^2} + \frac{1000}{(1+i)^3} \]

Therefore, solve for i (yield to maturity)

\[ (\text{price of bond}) = \text{present value} \quad \text{i.e.} \]
\[ P = \frac{300}{(1+i)} + \frac{300}{(1+i)^2} + \frac{1000}{(1+i)^3} \]

assuming
\[ P = 800, \ i \approx 0.34 \ (\text{YTM}) \]
\[ P = 1000, \ i \approx 0.22 \ (\text{YTM}) \]
\[ P = 1200, \ i \approx 0.12 \ (\text{YTM}) \]

*The YTM for a coupon bond depends inversely on the price*

**Vocabulary:**
- A bond priced at face value sells *at par*
- A bond priced below face value sells *at a discount*
- A bond priced above face value sells *at a premium*
Discount Bond

Pays face value after n years (zero coupon)

PDV of face value = \( \frac{FV}{(1+i)^n} \)

To calculate YTM, equate PDV with current price, solve out \( i \)

\[
P = \frac{FV}{(1+i)^n}
\]

... or ...

\[
\left( \frac{FV}{P} \right)^{\frac{1}{n}} - 1 = i'
\]

Example: let \( FV=1000, n=10 \)

\[
\Rightarrow
\]

\[
\begin{align*}
&\text{if } P=800, \quad i' = 0.022 \text{ (2.2\%)} \\
&\text{if } P=600, \quad i' = 0.052 \text{ (5.2\%)} \\
&\text{if } P=400, \quad i' = 0.095 \text{ (9.5\%)}
\end{align*}
\]
**Consols ("Perpetuities")**

Special case of a coupon bond with an eternal maturity

\[ PDV = \frac{C}{(1+i)} + \frac{C}{(1+i)^2} + \frac{C}{(1+i)^3} + \ldots + \frac{C}{(1+i)^\infty} \]

where \( C \equiv \) coupon payment

\[ PDV/C = (1+i)^{-1} + (1+i)^{-2} + (1+i)^{-3} + \ldots + (1+i)^{-\infty} \]

Note that the infinite series \( 1+x+x^2+x^3+\ldots x^\infty \) = \( 1/(1-x) \) for \( x < 1 \)

**proof:**

\[
\text{(Value of series)}(1-x) = (1-x)(1+x+x^2+x^3+\ldots x^\infty) = (1+x+x^2+x^3+\ldots x^\infty) - (x+x^2+x^3+\ldots x^\infty+1) = 1
\]

\[ \Rightarrow \text{(Value of series)} = (1/(1-x)) \]

Our series has this form if we add 1 to both sides

\[
(PDV/C)+1 = [1+(1+i)^{-1} + (1+i)^{-2} + (1+i)^{-3} + \ldots + (1+i)^{-\infty}] = i*[1+(1+i)^{-1} + (1+i)^{-2} + (1+i)^{-3} + \ldots + (1+i)^{-\infty}]
\]

\[ \text{ergo... (let } x=(1/(1+i))) \]

\[ (PDV/C)+1 = (1/(1-(1/(1+i)))) = (i+1)/i \]

\[ PDV/C = ((i+1)/i) - 1 = 1/i \]

\[ PDV = C/i \]

\[ \Rightarrow \text{The present discounted value of a consol is the coupon divided by the simple interest rate} \]
Other Measures of Interest Rates

Note: *YTM is the most accurate measure, though it is sometimes difficult to Calculate*

**Current Yield**

Approximates YTM for long bonds

The shorter the maturity, the worse the approximation

\[ i_c = \frac{C}{P}, \]

where

\[ i_c \equiv \text{current yield} \]
\[ P \equiv \text{price of bond} \]
\[ C \equiv \text{coupon} \]

inverting ...

\[ \frac{C}{i_c} = P \quad \text{(as in fair pricing of consols)} \]
Yield on a Discount Basis

Formula:

\[ i_{db} = \left( \frac{F-P}{F} \right) \times \frac{360}{(\text{days to maturity})} \]

where

- \( i_{db} \) = yield on a discount basis
- \( F \) = face value of discount bond
- \( P \) = purchase price of discount bond

Note: this measure exists for its computational simplicity Nevertheless, still in use (particularly in Treasuries)

It understates YTM b/c

- a) it uses a 360-day year
- b) it uses \( \% \Delta F \), not \( P \) (it isn't \( (P-F)/P \) ... ) This difference matters more the longer the maturity (since \( P \& F \) diverge over time)
The Distinction Between Interest Rates and Returns

The *rate of return* (RET) and the *yield to maturity* are two completely different things.

YTM assumes you hold the bond to maturity.

**What's the RET, if you sell before bond matures?**

It depends on the change in the bond's price.

- **Cost to owner**: purchase price of bond
- **Benefits to owner**: coupon payments + sale price of bond

For a bond held one year

$$RET = \frac{(P_{t+1} + C_t - P_t)}{P_t}$$

where

- $P_{t+1}$: Sale price of bond
- $C_t$: Coupon payments
- $P_t$: Purchase price of bond

or, broken out

$$RET = \frac{(P_{t+1} - P_t)}{P_t} + \frac{C_t}{P_t}$$

= rate of capital gain + current yield
### TABLE 2: One year returns on different maturity 10% coupon rate bonds purchased at par when the interest rate (YTM) rises from 10% to 20%

<table>
<thead>
<tr>
<th>Years to Maturity when purchased</th>
<th>Initial Current Yield (%)</th>
<th>Initial Price</th>
<th>Price Next Year (see coupon bond pricing formula: Equation 3 in text)</th>
<th>Rate of Capital Gain</th>
<th>Rate of Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>10</td>
<td>1000</td>
<td>503</td>
<td>-49.7</td>
<td>-39.7</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>1000</td>
<td>516</td>
<td>-48.4</td>
<td>-38.4</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>1000</td>
<td>597</td>
<td>-40.3</td>
<td>-30.3</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1000</td>
<td>741</td>
<td>-25.9</td>
<td>-15.9</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>1000</td>
<td>917</td>
<td>-8.3</td>
<td>1.7</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>1000</td>
<td>1000</td>
<td>0.0</td>
<td>10.0</td>
</tr>
</tbody>
</table>

Note how

1) Price inversely related to interest rate
2) Effect stronger, the longer the maturity of the bond

Prices and returns for long-term bonds are more volatile than those for shorter term bonds.

"Interest rate risk" (defined): the risk that fluctuating interest rates pose for the value of bond portfolios
The Distinction Between Real and Nominal Interest Rates

So far we have only discussed nominal interest rates

What matters for how people behave (savers and investors, that is) is not nominal but real interest rates

The Fisher equation:

\[ i_r = i - \pi^e \]

where

\( i_r \) real interest rate
\( i \) nominal interest rate
\( \pi^e \) expected inflation rate

When real interest rates are low
greater incentives to borrow
smaller incentives to lend

When real interest rates are high
smaller incentives to borrow
greater incentives to lend