CSci 1302 Rules and examples for deductive proofs in predicate logic.

In this course we use the following notations:

- $x_\forall$ stands for a *genuine* variable, i.e. a variable that stands for *any* element of the domain of the quantifier. Another way of saying it is that there are no assumptions made about this variable in the course of the proof. In rules "genuine" is abbreviated as "gen".
- $x_\exists$ stands for an *unknown* variable, i.e. a variable that stands for a fixed unknown element. Such variables usually are the result of $\exists E$ rule.
- Letters a, b, c denote constants - fixed elements of the domain. For example, constants may be specific numbers (-1, 2, 0, etc.) if your domain is $\mathbb{Z}$ (the set of all integers), or specific cities, such as London, if your domain is the set of all cities.

There are 8 rules, broken into 4 groups: the introduction (I) and elimination (E) rules for each of the two quantifiers ($\forall$ and $\exists$). The rules in the same group are additionally marked so that they can be distinguished from each other.

For instance, there is only one rule for eliminating $\exists$, so it's marked as $\exists E$, but there are three rules in $\exists I$, so $\exists I(const)$ specifies introduction of $\exists$ using a constant.

"New name" in $\exists E$ means that this variable name hasn't been used in this proof yet. In $\forall I$ rule all occurrences of $x_\forall$ must be repalced by x.

$$\frac{p(x_\forall)}{\therefore \forall x.p(x)} \quad \forall I(\text{all } x_\forall \text{ replaced}) \qquad \frac{\exists x.p(x)}{\therefore p(x_\exists)} \quad \exists E \ (x_\exists \text{ is a new name})$$

$$\frac{\forall x.p(x)}{\therefore p(a)} \quad \forall E(const) \qquad \frac{p(a)}{\therefore \exists x.p(x)} \quad \exists I(const)$$

$$\frac{\forall x.p(x)}{\therefore p(x_\forall)} \quad \forall E(gen) \qquad \frac{p(x_\forall)}{\therefore \exists x.p(x)} \quad \exists I(gen)$$

$$\frac{\forall x.p(x)}{\therefore p(x_\exists)} \quad \forall E(unknown) \qquad \frac{p(x_\exists)}{\therefore \exists x.p(x)} \quad \exists I(unknown)$$

You may also use equivalences that we have shown in predicate logic:

$\sim\forall x.p(x) \equiv \exists x.\sim p(x)$, $\sim\exists x.p(x) \equiv \forall x.\sim p(x)$ (negation of quantifiers).

Simple examples of proofs.

Example 1. We use a property that holds for all elements of the domain to prove a property for a specific constant element.

$$\begin{array}{ll} 1. & \forall x.isEven(x) \rightarrow isDivisible(x,2) \\ 2. & isEven(6) \\ & \overline{\qquad\qquad} \\ & \therefore isDivisible(6,2) \end{array}$$

Proof:

$$\begin{array}{lll} 3. & isEven(6) \rightarrow isDivisible(6,2) & 1, \forall E(const) \\ 4. & isDivisible(6,2) & 2, 3, MP. \end{array}$$

Example 2. Here we know that there is some number that's even, so we can conclude that there is a number that's divisible by 2. However, since we are not given a specific even number, at the end we don't know what number is divisible by 2. The number is denoted by an "unknown" variable $x_\exists$.

$$\begin{array}{ll} 1. & \forall x.isEven(x) \rightarrow isDivisible(x,2) \\ 2. & \exists x.isEven(x) \\ & \overline{\qquad\qquad} \\ & \therefore \exists x.isDivisible(x,2) \end{array}$$

Proof:

$$\begin{array}{lll} 3. & isEven(x_\exists) & 2, \exists E \\ 4. & isEven(x_\exists) \rightarrow isDivisible(x_\exists,2) & 1, \forall E(unknown) \\ 5. & isDivisible(x_\exists,2) & 3, 4, MP \\ 6. & \exists x.isDivisible(x,2) & \end{array}$$

Note that we must start with $\exists E$ rule because it requires a new variable. If I start with eliminating $\forall$ first, I get $isEven(x_\exists) \rightarrow isDivisible(x_\exists, 2)$, but then I can't use $\exists E$ for $x_\exists$ because this name was already used in the proof. **Tip:** eliminate $\exists$ quantifiers before $\forall$.

Example 3. The intuition behind the last step is that since 10 is divisible by 1 and 5, there is at least one number that's divisible by 1 and 5.

$$\begin{array}{ll} 1. & \forall x.isDivisible(x,1) \\ 2. & isDivisible(10,5) \\ & \overline{\qquad\qquad} \\ & \therefore \exists x.isDivisible(x,1) \wedge isDivisible(x,5) \end{array}$$

Proof:

$$\begin{array}{lll} 3. & isDivisible(10,1) & 1, \forall E(const) \\ 4. & isDivisible(10,1) \wedge isDivisible(10,5) & 2, 3, \wedge I \\ 5. & \exists x.isDivisible(x,1) \wedge isDivisible(x,5) & 4, \exists I(const) \end{array}$$

Example 4. Please study this proof carefully and understand why all the rules are followed and the proof is valid.

1.	$\forall x.isDivisible(x, 1)$
	———
	$\therefore \exists x.isDivisible(x, x)$

Proof:

2.	$isDivisible(1, 1)$	$1, \forall E(const)$
3.	$\exists x.isDivisible(x, x)$	$2, \exists I(const)$