CSci 1302 Example of a proof in predicate logic

The example uses the following predicates on integer numbers:

1. \( \text{next}(x,y) \) is true if \( y = x + 1 \), false otherwise.

2. \( \text{greater}(x,y) \) is true if \( x > y \), false otherwise.

Below are two proofs of the same mathematical argument: if every number has the next number and the next number of \( x \) is greater than \( x \), then there is no greatest number. Note that we need to add to our list of assumptions the property that if \( x \) is greater than \( y \), then \( y \) is not greater than \( x \). Without this assumption the proof does not go through.

There are two versions of the proof. The first one uses the theorems of predicate logic \( \lnot \exists x. p(x) \equiv \forall x. \neg p(x) \) and \( \forall x. p(x) \equiv \exists x. p(x) \). The other one is a proof by contradiction.

1. \( \forall x. \exists y. \text{next}(x,y) \) \hspace{1cm} Assumption
2. \( \forall x. \forall y. \text{next}(x,y) \rightarrow \text{greater}(y,x) \) \hspace{1cm} Assumption
3. \( \forall x. \forall y. \text{next}(x,y) \rightarrow \neg \text{greater}(y,x) \) \hspace{1cm} Assumption

\[ \vdash \lnot \exists x \forall y. \text{greater}(x,y) \]

4. \( \exists y. \text{next}(x_0, y) \) \hspace{1cm} 1, \( \forall \exists \) \( \text{gen} \), \( y \) depends on \( x \)
5. \( \text{next}(x_0, y) \) \hspace{1cm} 4, \( \exists \exists \)
6. \( \text{next}(x_0, y) \rightarrow \text{greater}(y, x_0) \) \hspace{1cm} 2, \( \forall \exists \) \( \text{unknown} \) twice
7. \( \text{greater}(y, x_0) \) \hspace{1cm} \( x_0 \) appeared before, so it’s unknown, not genuine
8. \( \text{greater}(y_1, x_0) \rightarrow \neg \text{greater}(x_0, y_1) \) \hspace{1cm} 3, \( \forall \exists \) \( \text{unknown} \) twice
9. \( \neg \text{greater}(x_0, y_1) \) \hspace{1cm} 7, 8, Modus Ponens
10. \( \exists y. \text{greater}(x_0, y) \) \hspace{1cm} 9, \( \exists \exists \)
11. \( \forall x. \exists y. \text{greater}(x, y) \) \hspace{1cm} 10, \( \forall \exists \)
12. \( \forall x. \forall y. \text{greater}(x, y) \) \hspace{1cm} 11, Thm. \( \lnot \forall \forall \equiv \exists \exists \)
13. \( \lnot \exists x. \forall y. \text{greater}(x, y) \) \hspace{1cm} 12, Thm. \( \lnot \exists \forall \equiv \forall \exists \)

Note that step 10 introduces \( \exists \) quantifier for a variable \( y_1 \) which depends on \( x_0 \). This is OK. However, introducing the \( \forall \) quantifier would not have been allowed.
The same example can be done using a proof by contradiction: assume that there is the greatest number, and find a contradiction.

1. $\forall x. \exists y. \text{next}(x, y)$  
   \textit{Assumption}
2. $\forall x. \forall y. \text{next}(x, y) \rightarrow \text{greater}(y, x)$  
   \textit{Assumption}
3. $\forall x. \forall y. \text{greater}(x, y) \rightarrow \neg \text{greater}(y, x)$  
   \textit{Assumption}

   $\neg \exists x. \forall y. \text{greater}(x, y)$
4. $\exists x. \forall y. \text{greater}(x, y)$  
   \textit{Assumption}(there is the largest number)
5. $\forall y. \text{greater}(x_\exists, y)$  
   4, $\exists \exists \text{E}$
6. $\exists y. \text{next}(x_\exists, y)$  
   1, $\forall \text{E}(unknown)$
7. $\text{next}(x_\exists, y_\exists)$  
   6, $\exists \exists \text{E}$
8. $\text{next}(x_\exists, y_\exists) \rightarrow \text{greater}(y_\exists, x_\exists)$  
   7, $\forall \text{E}(unknown)$ twice
9. $\text{greater}(y_\exists, x_\exists)$  
   7, 8, Modus Ponens
10. $\text{greater}(y_\exists, x_\exists) \rightarrow \neg \text{greater}(x_\exists, y_\exists)$  
    3, $\forall \text{E}(unknown)$ twice
11. $\neg \text{greater}(x_\exists, y_\exists)$  
    9, 10, Modus Ponens
12. $\text{greater}(x_\exists, y_\exists)$  
    5, $\forall \text{E}(unknown)$
13. $\neg \exists x. \forall y. \text{greater}(x, y)$  
    4 – 12, Contradiction lines 11, 12