CSci 1302 Assignment 11

Due Wedn., Dec. 15th 2004

Notations: $\emptyset$ stands for the empty set, $\mathbb{N}$ is the set of natural numbers $(1, 2, 3, ...)$. $\mathbb{P}X$ stands for the power set of X.

Problem 1 (6 points). For each of the relations below:

- Write at least 5 elements that belong to the relation. If the relation has fewer than 5 elements, then write all of them.
- Find $dom(R)$.
- Find $ran(R)$.

The relations for the problem are as follows (all are considered on $\mathbb{N} \times \mathbb{N}$):

1. $R = \{(n, m) \mid n = m^2\}$
2. $R = \{(n, m) \mid n \text{ is divisible by } m, m \neq 1, m \neq n\}$
3. $R = \{(n, m) \mid n^2 + m^2 = 3\}$
4. $R = \{(n, m) \mid n^2 + m^2 = 5\}$

Problem 2 (6 points). Given a set $A = \{a, b, c, e, f\}$, a set $B = \{w, x, y, z\}$, and the relations

$$S_1 = \{(a, x), (b, x), (c, y), (c, z), (f, z)\},$$
$$S_2 = \{(a, y), (b, x), (c, y), (c, w), (e, z), (e, x)\}$$

do the following:

1. for each of S_1, S_2 find domain, range. Find $S_1^{\sim}$, i.e. the inverse relation of S_1,
2. find $S_1 \cap S_2$ and $S_1 \cup S_2$.
3. Draw the graphs of both relations in such a way that makes it easy to compute their composition (see Figure 14.1 on p. 213 for an example).
4. Compute composition $R; S$.

Problem 3 (10 points). You are given three sets $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, and $C = \{red, white, blue\}$ and three relations:

- $R : A \times C = \{(a, red), (b, blue), (a, white)\}$,
- $S : B \times C = \{(1, blue), (3, red)\}$,
- $T : A \times A = \{(a, a), (b, c), (c, b)\}$

For each of the following operations on relations please state whether the operation makes sense. If it makes sense, then compute the result. If it doesn't, then explain why.

1. $R; S$
2. $T; R$
3. $R; T$
4. $R^{\sim}; T$
5. $R^{\sim}; T^{\sim}$
6. $R; S^{\sim}$
7. $T; T^{\sim}; R$
8. R^2
9. T^2
10. T^0

Problem 4 (6 points). Please classify the following relations on natural numbers $\mathbb{N}$ as:
reflexive/irreflexive/non-reflexive,
symmetric/anti-symmetric/non-symmetric, and
transitive/non-transitive.

1. $R = \{(n, m) \mid n + m \text{ is even}\}$
2. $R = \{(n, m) \mid n + m \text{ is odd}\}$
3. $R = \{(n, m) \mid n \text{ is even}, m \text{ is even}\}$
4. $R = \{(n, m) \mid n \text{ is even}, m \text{ is odd}\}$ (think carefully about transitivity of this relation).

Problem 5 (6 points). Construct a reflexive, a symmetric, a transitive (R^+), and a reflexive transitive (R^*) closures of each of the following relations on the set $A = \{a, b, c, d\}$. You may represent the resulting relation as a picture instead of writing out the list of its elements.

1. $R = \{(a, b), (b, c), (c, d), (d, a)\}$,
2. $R = \{(a, a), (a, b), (a, d), (c, c), (c, b), (c, d)\}$.

Problem 6 (4 points). Consider the following directed graph: $V = \{a, b, c, d\}$, $E = \{(a, b), (b, a), (b, b), (b, c), (b, d), (c, d)\}$.

1. draw the picture of the graph
2. write down the adjacency matrix of the graph
3. find all simple paths from a to c
4. find all cycles in the graph

Problem 7 (4 points). Consider the following undirected graph: $V = \{a, b, c, d\}$, $E = \{(a, b), (b, b), (b, c), (b, d), (c, d)\}$.

1. draw the picture of the graph
2. write down the adjacency matrix of the graph
3. find all simple paths from a to c
4. find all cycles in the graph

Problem 8 (5 points). Consider the following relations from the set $A = \{a, b, c, d\}$ to the set $B = \{1, 2, 3, 4\}$:

1. $\{(a, 1), (b, 1), (c, 1)\}$
2. $\{(a, 1), (a, 2), (a, 3)\}$
3. $\{(a, 1), (b, 2), (c, 3), (d, 4), (a, 2)\}$
4. $\{(a, 1), (b, 2), (c, 3), (d, 1)\}$
5. $\{(a, 1), (b, 2), (c, 1), (d, 2)\}$

For each relation please say whether the relation is a function, and if it is, whether it is partial or total. Please explain your answer briefly.

Problem 9 (Extra Credit, 5 points) Let R be a relation. Consider the following two relations:

1. Suppose that R_r is the reflexive closure of R, R_s is the symmetric closure of R, and R^+ is the (non-reflexive) transitive closure of R. Let $\widehat{R} = R_r \cup R_s \cup R^+$.
2. Suppose that R_r is the reflexive closure of R, R'_s is the symmetric closure of R_r (i.e. of the reflexive closure of R, not of R itself), and $\overline{R}$ is the (non-reflexive) transitive closure of R'_s.

Are the two relations ($\widehat{R}$ and $\overline{R}$) equal for all possible relations R? Is any one of them guaranteed to be an equivalence relation? In case of a positive answer please justify, in case of a negative answer please give a counterexample.

You may earn a partial credit on this problem if you answer some, but not all, of the questions.