GALOIS NUMBER FIELDS WITH SMALL ROOT DISCRIMINANT

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Abstract. We pose the problem of identifying the set $\mathcal{K}(G,\Omega)$ of Galois number fields with given Galois group G and root discriminant less than the Serre constant $\Omega \approx 44.7632$. We definitively treat the cases $G = A_4$, A_5, A_6 and S_4, S_5, S_6, finding exactly 59, 78, 5 and 527, 192, 13 fields respectively. We present other fields with Galois group $SL_3(2)$, A_7, S_7, $PGL_2(7)$, $SL_2(8)$, $\Sigma L_2(8)$, $PGL_2(9)$, $P\Gamma L_2(9)$, $PSL_2(11)$, and $A_5^2.2$, and root discriminant less than Ω. We conjecture that for all but finitely many groups G, the set $\mathcal{K}(G,\Omega)$ is empty.

1. Introduction

There is a large literature on number fields with small absolute discriminant. Most of this literature is focused on number fields of small degree. A standard resource summarizing much of this literature is [2], which tabulates results in degrees ≤ 7. In this paper, we focus instead on Galois number fields. Our fields have relatively large degrees, but they are still accessible via their low degree subfields.

Fix a finite group G. Let $\mathcal{K}(G)$ be the set of Galois number fields $K \subset \mathbf{C}$ with $\mathrm{Gal}(K/\mathbf{Q})$ isomorphic to G. For $C \in [1,\infty)$, let $\mathcal{K}(G,C) \subseteq \mathcal{K}(G)$ be the subset of fields with root discriminant $\leq C$. It is a classical theorem that all $\mathcal{K}(G,C)$ are finite.

For a given G, a natural computational problem is to explicitly produce $\mathcal{K}(G,C)$ for C as large as possible. This is the sort of information presented at [2] in the context of low degree fields. To keep the focus as much as possible on theory, rather than on very long tables, we work here with a single rather small cutoff $\Omega = 8\pi e^{\gamma} \approx 44.7632$. Our main problem, for a given group G, thus becomes to determine $\mathcal{K}(G,\Omega)$. Also of interest for us is d_G, the smallest root discriminant of any field in $\mathcal{K}(G)$.

The constant Ω was introduced first by Serre in 1975 [22]; see also [19]. It is an interesting cutoff for the following reason. Let $\mathcal{K}_{\mathrm{all}}(C)$ be the set all number fields in $\mathbf{C}$ with root discriminant $\leq C$. Let $\mathcal{K}(C) \subseteq \mathcal{K}_{\mathrm{all}}(C)$ be the subset of Galois fields, so that $\mathcal{K}(C) = \cup_G \mathcal{K}(G,C)$. Then it is known that $\mathcal{K}_{\mathrm{all}}(C)$ is finite for $C < \Omega/2$, and that the Generalized Riemann Hypothesis implies that $\mathcal{K}_{\mathrm{all}}(C)$ is finite for $C < \Omega$ as well. So, of course, $\mathcal{K}(C)$ is finite for $C < \Omega/2$ and conditionally finite for $C < \Omega$ too.

We have tried to take our computational results far enough so as to get some feel for how the answer to our main problem looks for general G. First of all, we expect that our cutoff is indeed extremely low in the following sense:

Conjecture 1.1. *$\mathcal{K}(G,\Omega)$ is empty for all but finitely many groups.*

This conjecture is plausible just from the theoretical discussion in the preceding paragraph, as indeed the Generalized Riemann Hypothesis would imply it is true with Ω replaced by any smaller number. The real general import of our computations is that the sets $\mathcal{K}(G,\Omega)$ seem to be quite small indeed. It seems even possible that the only non-abelian simple groups involved in a G with nonempty $\mathcal{K}(G,\Omega)$ are the five smallest,

$$\begin{array}{rclcll} SL_2(4) & \cong & PSL_2(5) & \cong & A_5, & \text{(see §5,12)} \\ PSL_2(7) & \cong & SL_3(2), & & & \text{(see §7,8)} \\ PSL_2(9) & \cong & A_6, & & & \text{(see §5,10)} \\ SL_2(8), & & & & & \text{(see §9)} \\ PSL_2(11), & & & & & \text{(see §11)} \end{array}$$

and the eighth-smallest, A_7 (see §7). These simple groups have orders 60, 168, 360, 504, 660, and 2520 respectively.

Computing root discriminants in our Galois context is substantially harder than computing root discriminants in the traditional low degree setting. Section 2 sketches how we do this, and details are given in [12].

Section 3 determines $\mathcal{K}(G,\Omega)$ for abelian groups G, where one can make use of the Kronecker-Weber theorem. We find in particular that there are exactly 7063 abelian fields with root discriminant $\leq \Omega$, the one of largest degree being $\mathbf{Q}(e^{2\pi i/77})$ of degree sixty with with root discriminant $7^{5/6}11^{9/10} \approx 43.80$. In Section 4, we consider the class fields of these 7063 abelian fields, getting more elements of $\mathcal{K}(\Omega)$, of which the largest has degree $2^{12} \cdot 3 \cdot 7 = 86{,}016$ with root discriminant $3^{1/2}29^{27/28} \approx 44.54$.

Section 5 determines $\mathcal{K}(G,\Omega)$ for most groups G embeddable in S_6. We find that for $G = A_4$, A_5, A_6, and S_4, S_5, S_6, there are exactly 59, 78, 5, and 527, 192, 13 fields in $\mathcal{K}(G,\Omega)$.

In the remaining sections, we restrict attention to groups G of the form $H^m.A$ for H a non-abelian simple group, m a positive integer, and A a subgroup of the outer automorphism group of H^m. In fact, $m = 1$ in all our examples except for the case A_5 where we consider also $m = 2$. We make this restriction only to keep the current paper within reasonable bounds.

Section 6 describes how one can use holomorphic modular forms on the upper half plane as well as less classical automorphic forms to search for sub-Ω fields. We revisit the A_6 fields from the previous section as examples. In the next four sections, we will indicate how this automorphic approach complements our direct search for defining equations.

Section 7 treats the three nonsolvable groups which can be realized as transitive subgroups of S_7. We find seventeen fields in $\mathcal{K}(SL_3(2),\Omega)$ and one field in $\mathcal{K}(A_7,\Omega)$ the lists being complete out through root discriminant 39.52. For $G = S_7$, we know only $|\mathcal{K}(G,\Omega)| \geq 1$ with our best field showing $d_{S_7} \leq 2^{3/2}3^{7/6}5^{6/7} \approx 40.49$.

Sections 8, 9, 10, and 11 present sub-Ω fields for certain nonsolvable subgroups of S_8, S_9, S_{10}, and S_{11} respectively, namely $PGL(7)$, $SL_2(8)$, $\Sigma L_2(8)$, $PGL_2(9)$, $P\Gamma L_2(9)$, and $PSL_2(11)$. For $SL_2(8)$ we use the modular approach. For the remaining groups, our main method is to suitably specialize three point covers.

Section 12 uses the lists from §5 to prove that $\mathcal{K}(G,\Omega)$ is empty for certain G, for example $G = A_5 \times A_5$. In contrast, it reports on a field K with $\mathrm{Gal}(K/\mathbf{Q}) \cong A_5^2.2$ and root discriminant $2^{51/16}3^{15/18} \approx 41.90$.

Section 13 reports on our efforts to find small root discriminant Galois fields for other non-solvable groups. While we naturally obtained upper bounds for d_G, we did not find any more sub-Ω fields. We illustrate the general nature of these searches by treating the case $G = \Sigma L_2(16)$ in some detail. In this case, we specialize two three point covers and find $d_{\Sigma L_2(16)} \leq 2^{101/60} 3^{3/4} 5^{23/20} \approx 46.60$.

The computations behind this paper made extensive use of the *Pari* library [20]. Also we made substantial use of the ATLAS [6] as a source of group-theoretical facts. To remove potential ambiguities, we use the "T notation" for transitive permutation groups as well as descriptive notation. This T notation was introduced in [3] and is used also in [20] and [14].

The website [9] is a companion to this paper. For various G, it gives the currently known fields in $\mathcal{K}(G, \Omega)$ and indicates through what cutoff the list is known to be complete. We plan to consider G beyond those discussed in this paper, and place our findings on this website.

2. Computing Galois root discriminants via slope data

Let $f(x) \in \mathbf{Z}[x]$ be a monic irreducible degree n polynomial. Consider the abstract field $F = \mathbf{Q}[x]/f(x)$ and also the splitting field $K \subset \mathbf{C}$ associated to f. The abstract field F is typically non-Galois and K is the Galois closure of any embedding of F in $\mathbf{C}$. Let $N = [K : \mathbf{Q}]$ so that $N = |\operatorname{Gal}(K/\mathbf{Q})|$. Then n divides N, which in turn divides $n!$.

Let $D(F)$, $D(K) \in \mathbf{Z}$ be the corresponding field discriminants. Let $d(F) = |D(F)|^{1/n}$ and $d(K) = |D(K)|^{1/N}$ be the corresponding root discriminants. The main quantity for us is $d(K)$. We call it the Galois root discriminant of f, F, or K, and abbreviate "Galois root discriminant" by GRD.

One has

$$d(F) \leq d(K),$$

with equality iff $K/\sigma(F)$ is unramified for one or equivalently any embedding $\sigma : F \to \mathbf{C}$. More sharply, one has canonical factorizations

$$d(F) = \prod_p p^{\alpha_p}, \tag{2.1}$$

$$d(K) = \prod_p p^{\beta_p}, \tag{2.2}$$

with all exponents rational numbers. One has

$$\alpha_p \leq \beta_p, \tag{2.3}$$

with equality iff $K/\sigma(F)$ as above is unramified at p.

To compute the Galois root discriminant $d(K)$, we work one prime at a time, computing each of the β_p. The p-adic computation of β_p never sees the typically large globally-defined number N. If p is tamely ramified, the computation is relatively easy, going as follows. Let $\mathbf{Q}_p^{\mathrm{un}}$ be the maximal unramified extension of $\mathbf{Q}_p$. Factor f over $\mathbf{Q}_p^{\mathrm{un}}$ and let e_1, e_2, $\ldots$, e_g be the degrees of the factors. We call $(e_1, \ldots, e_g)$ the ramification partition of f at p. Let t be the least common multiple of the e_i's. Then

$$\beta_p = 1 - \frac{1}{t}. \tag{2.4}$$

For comparison,

$$\alpha_p = \frac{1}{n}\sum_{i=1}^{g}(e_i - 1).$$

So equality holds in (2.3) iff all the e_i coincide. Of course, for all but finitely many p, all the e_i are 1 and $\alpha_p = \beta_p = 0$.

When p is wildly ramified, the computation of β_p is much harder, and focuses on wild slopes. Suppose the p-inertia group I_p has order $p^k t$ with t prime to p. Then one has k wild slopes at p, which we always index in ascending order,

$$s_1 \leq s_2 \leq \cdots \leq s_{k-1} \leq s_k,$$

all wild slopes being rational numbers greater than one. One then has

$$\beta_p = \left(\sum_{j=1}^{k} \frac{p-1}{p^{k+1-j}} s_j\right) + \frac{1}{p^k}\frac{t-1}{t} \tag{2.5}$$

with $\beta_p \geq 1$ iff $k \geq 1$. When describing a given Galois number field K we often give the slope data $p[s_1, \cdots, s_k]_t$, rather than just the number β_p. When p is tame, so that there are no wild slopes, we simply write p_t. Let t_0 be the prime-to-p part of the least common multiple of the denominators of the s_k. Always t_0 divides t. When t_0 is equal to t, a very common occurence in the setting of this paper, we typically omit the subscript t from the notation. Thus $2[4/3, 4/3]_3$ is abbreviated $2[4/3, 4/3]$ and $3[3/2]_2$ is abbreviated $3[3/2]$. For a discussion of how one goes about computing the s_j, and for web-based software which carries out such computations, see [12].

If 2 divides the discriminant of a number field, then it contributes at least $2^{2/3}$ to the field's Galois root discriminant. Any odd prime p dividing the discriminant of a number field contributes at least $p^{1/2}$. So one immediately has restrictions on the set S of primes dividing the discriminant of a field K in $\mathcal{K}(\Omega)$. If S is empty, then $K = \mathbf{Q}$. If $S = \{p\}$, then p must be one of the 304 primes ≤ 2003 as $\Omega^2 \approx 2003.75$. For $S = \{p, q\}$, $\{p, q, r\}$, and $\{p, q, r, s\}$, there are respectively 533, 264, and 36 possiblities. Already $S = \{p, q, r, s, t\}$ is impossible as $2^{2/3}(3 \cdot 5 \cdot 7 \cdot 11)^{1/2} \approx 45.32$.

The 1137 possible non-empty S come in two types. For some, wild ramification is locally possible, and then there are infinitely many locally possible GRDs less than Ω. For the rest, only tame ramification is possible and then there are only finitely many possible GRDs. The case $S = \{p\}$ is of the first type exactly for the 14 primes $p \leq 43$. For $S = \{p, q\}$, $\{p, q, r\}$, and $\{p, q, r, s\}$, there are respectively 167, 116, and 16 possibilities where wild ramification is allowed. Of course if a group G is fixed, only some slopes are possible, and one has only finitely many locally possible sub-Ω GRDs. The case $G = S_6$ is explained in more detail in Section 5.

3. Abelian fields

To get started, we consider abelian fields. This section explains the proof of the following proposition.

Proposition 3.1. *For G an abelian group, $|\mathcal{K}(G, \Omega)|$ is as given in Table 3.1. In particular, only 59 abelian groups G have $|\mathcal{K}(G, \Omega)| > 0$ and altogether there are* 7063 *abelian fields in $\mathcal{K}(\Omega)$.*

We use the standard complete classification of abelian fields in terms of their ramification. Thus in this section, we do not need to work with the fields themselves.

For r in the group $\mathbf{Q}/\mathbf{Z}$, let ζ_r be the root of unity $\exp(2\pi i r)$. The field $\mathbf{Q}^{\mathrm{ab}}$ generated by these complex numbers is known to be the maximum abelian extension of $\mathbf{Q}$ in $\mathbf{C}$. So $\mathbf{Q}^{\mathrm{ab}}$ is the union of the subfields $\mathbf{Q}(\zeta_{1/n})$

Let $\hat{\mathbf{Z}}$ be the profinite completion of the integers $\mathbf{Z}$, so that $\hat{\mathbf{Z}} = \varprojlim \mathbf{Z}/n$. A unit $a \in \hat{\mathbf{Z}}^\times = \varprojlim (\mathbf{Z}/n)^\times$ acts on $\mathbf{Q}^{\mathrm{ab}}$ via $\sigma_a(\zeta_r) = \zeta_{ar}$. Via this action one has an isomorphism $\hat{\mathbf{Z}}^\times \xrightarrow{\sim} \mathrm{Gal}(\mathbf{Q}^{\mathrm{ab}}/\mathbf{Q}) : a \mapsto \sigma_a$.

The ring $\hat{\mathbf{Z}}$ factors canonically into a product over all primes p of the p-adic integers $\mathbf{Z}_p$, with $\mathbf{Z}_p = \varprojlim \mathbf{Z}/p^j$. On the level of unit groups, this becomes

$$\hat{\mathbf{Z}}^\times = \prod_p \mathbf{Z}_p^\times. \tag{3.1}$$

Let T_p be the torsion subgroup of $\mathbf{Z}_p^\times$. For p odd, reduction modulo p gives an isomorphism $T_p \to \mathbf{F}_p^\times$ and so $|T_p| = p-1$; also $T_2 = \{1, -1\}$ and so $|T_2| = 2$. For p odd, let P_p be the group of principal units $1 + p\mathbf{Z}_p$ and define also $P_2 = 1 + 4\mathbf{Z}_2$. Then in all cases

$$\mathbf{Z}_p^\times = T_p \times P_p. \tag{3.2}$$

Quotient groups of T_p are indexed by their cardinality, which can be any divisor of $|T_p|$. Finite quotient groups of $|P_p|$ are also indexed by their cardinality, the possibilities being 1, p, p^2, ... For p odd, quotients of $\mathbf{Z}_p^\times$ are just the products of a quotient of T_p and a quotient of P_p; we let $I_{p,c}$ denote the unique quotient of cardinality c. For $p = 2$, we use the following notation for quotients of $\mathbf{Z}_2^\times$. Let $u = -1$ be the generator for T_2 and work with $v = 5$ as a generator for P_2. Then for $c = 2, 4, 8, \ldots$, we let

$$\begin{aligned} I_{2,c+} &= \mathbf{Z}_2^\times/\langle u, v^c\rangle, \\ I_{2,2,c/2} &= \mathbf{Z}_2^\times/\langle v^{c/2}\rangle, \\ I_{2,c-} &= \mathbf{Z}_2^\times/\langle (uv)^{c/2}\rangle. \end{aligned}$$

Also we let $I_{2,1}$ be the 1-element quotient of $\mathbf{Z}_2^\times$ and abbreviate $I_{2,2,1}$ by $I_{2,2}$.

In the factorization (3.1), $\mathbf{Z}_p^\times$ is exactly the inertia subgroup I_p. For p odd, T_p is identified with the tame quotient, and so is associated with the slope 1. P_p is thus the wild inertia subgroup, and its associated slopes are $j+1$ as j runs over the positive integers, each slope appearing with multiplicity 1. All of $I_2 = \mathbf{Z}_2^\times$ is wild, the slopes being 2, 3, 4, ... For the quotient $I_{2,2,2^{j-1}}$, the slopes are 2, ..., $j+1$ while for the quotients $I_{2,2^j+}$ and $I_{2,2^j-}$, the slopes are 3, ..., $j+2$.

To find all abelian fields $\neq \mathbf{Q}$ with root discriminant less than Ω we proceed in three stages. First, we restrict attention to fields ramified at exactly one prime. For p odd, these are the K_{p,tp^j} such that

$$\beta_p = \frac{1}{p^j}\frac{t-1}{t} + \sum_{i=1}^{j} \frac{p-1}{p^{j+1-i}}\,(i+1) \tag{3.3}$$

Table 3.1. Information about the 7062 abelian fields different from $\mathbf{Q}$ having root discriminant $\leq \Omega$.

$\|G\|$	G	#	$K_{G,1}$	d_G	$\|G\|$	G	#	$K_{G,1}$	d_G
2	2	1220	3_2	1.73	22	22	7	23_{22}	19.94
3	3	47	7_3	3.66	23	23	1	47_{23}	39.76
4	4	228	5_4	3.34	24	24	1	$7_3 17_8$	43.66
4	$2,2$	2421	$2_2 3_2$	3.46	24	$12,2$	73	$5_4 7_6$	16.92
5	5	7	11_5	6.81	24	$6,2,2$	70	$2_2 3_2 7_6$	17.53
6	6	399	7_6	5.06	28	28	2	29_{28}	25.71
7	7	4	29_7	17.93	28	$14,2$	1	$3_2 29_{14}$	39.49
8	8	23	17_8	11.93	30	30	8	31_{30}	27.65
8	$4,2$	581	$3_2 5_4$	5.79	32	$8,4$	1	$5_4 17_8$	39.89
8	$2,2,2$	908	$2_{2,2} 3_2$	6.93	32	$16,2$	7	$3_2 17_{16}$	24.67
9	9	3	19_9	13.70	32	$4,4,2$	4	$2_{2,4} 5_4$	26.75
9	$3,3$	9	$3_3 7_3$	15.83	32	$8,2,2$	4	$2_{2,8} 3_2$	27.71
10	10	69	11_{10}	8.65	32	$4,2,2,2$	10	$2_{2,2} 3_2 5_4$	23.17
11	11	1	23_{11}	17.30	32	$2,2,2,2,2$	1	$2_{2,2} 3_2 5_2 7_2$	40.99
12	12	66	13_{12}	10.50	36	36	1	37_{36}	33.47
12	$6,2$	391	$3_2 7_6$	8.77	36	$6,6$	6	$3_6 7_6$	26.30
13	13	1	53_{13}	39.05	36	$12,3$	1	$7_3 13_{12}$	38.42
14	14	8	29_{14}	22.80	36	$18,2$	9	$3_2 19_{18}$	27.94
15	15	4	31_{15}	24.66	40	40	1	41_{40}	37.36
16	16	9	17_{16}	14.24	40	$20,2$	5	$5_4 11_{10}$	28.94
16	$4,4$	16	$2_{4\pm} 5_4$	22.49	40	$10,2,2$	7	$2_2 3_2 11_{10}$	29.98
16	$8,2$	30	$2_{2,8}$	16.00	42	42	2	7_{42}	35.43
16	$4,2,2$	195	$2_2 3_2 5_4$	11.58	44	$22,2$	3	$3_2 23_{22}$	34.55
16	$2,2,2,2$	73	$2_{2,2} 3_2 5_2$	15.49	46	46	1	47_{46}	43.23
18	18	24	3_{18}	15.59	48	$12,4$	1	$5_4 13_{12}$	35.10
18	$6,3$	19	$3_6 7_3$	19.01	48	$12,2,2$	9	$3_2 5_4 7_6$	29.31
20	20	8	5_{20}	16.72	48	$6,2,2,2$	2	$2_{2,2} 3_2 7_6$	35.06
20	$10,2$	56	$3_2 11_{10}$	14.99	56	$28,2$	1	$3_2 29_{28}$	44.54
21	21	2	7_{21}	33.82	60	$30,2$	1	$7_6 11_{10}$	43.80

is less than $\log_p(\Omega)$. For $p = 2$, we have

$$\begin{aligned} K_{2,2+} &= \mathbf{Q}(\sqrt{2}), \\ K_{2,2} &= \mathbf{Q}(i), \\ K_{2,2-} &= \mathbf{Q}(\sqrt{-2}). \end{aligned}$$

For $d \geq 2$ a power of 2, we let $K_{2,2,d}$ be the full cyclotomic field $\mathbf{Q}(\zeta_{1/4d})$, with non-cyclic Galois group isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/d$. The remaining cases have cyclic Galois group, with $K_{2,c+}$ being the totally real field $\mathbf{Q}(\zeta_{4c})^+$ and $K_{2,c-}$ being totally imaginary. Here one has

$$\beta_2(K_{2,2,2^{j-1}}) = \frac{1}{2}(j+1) + \frac{1}{4}(j) + \cdots + \frac{1}{2^j}2 = j, \tag{3.4}$$

$$\beta_2(K_{2,2^j\pm}) = \frac{1}{2}(j+2) + \frac{1}{4}(j+1) + \cdots + \frac{1}{2^j}3 = j+1-\frac{1}{2^j}. \tag{3.5}$$

One finds that there are exactly 421 abelian fields ramified at exactly one prime.

Second, we take composita of the one-prime fields obtaining fields with root discriminant $\prod d(K_{p,c_p})$. Here c_p is a cardinality, augmented perhaps with a sign if $p=2$. One gets 1785 more fields. Finally, one looks within each composed field K_c to find proper subfields K with $d(K) = d(K_c)$. This is a somewhat intricate computation with abelian groups.

Table 3.1 summarizes our calculations. The abelian group G is given by its invariant factors, so that e.g. $6, 2$ represents the group $C_6 \times C_2$. The column # gives the number $|\mathcal{K}(G,\Omega)|$. The column $K_{G,1}$ gives the fields with the minimal root discriminant d_G, with a field $K_{p,c}$ being represented by the symbol p_c. So, for example, $2_{2,4}5_4$ stands for the compositum $K_{2,2,4}K_{5,4}$, which in turn is $\mathbf{Q}(\zeta_{1/16}, \zeta_{1/5})$. With this convention, if $K_{G,1}$ is tame then we are printing exactly its slope data. Note also that our conventions allow one to recover G from the subscripts in the $K_{G,1}$ column.

4. Class fields of abelian fields

Suppose K is a number field with narrow class number h. Then the narrow Hilbert class field H of K is an unramified degree h extension of K, so that H and K have the same root discriminant. In particular, if $K \in \mathcal{K}(\Omega)$ then $H \in \mathcal{K}(\Omega)$ too.

The fields in the previous section are all small enough so that we are able to compute their narrow class numbers. In contrast to the previous section, our computations here actually involve fields.

Figure 4.1 plots the pairs $(d(H), \log_{10}[H:\mathbf{Q}])$ for the resulting narrow Hilbert class fields. Its purpose is to give one a first intuitive feel for the set $\mathcal{K}(\Omega)$. For example, from the previous section we know that the primes 2, 3, 5, 7, 11, 13, and 23, can divide the degree of a field in $\mathcal{K}(\Omega)$. From the computations behind this section, we also know that 17, 19, 29, 31, 37, 41, 43, 61, 67, 73, 89, 97, 109, 139, 151, 163, 211, 271, and 331 can divide the degree of a field in $\mathcal{K}(\Omega)$. Sections 5–12 do not produce any more such primes; in fact, they only reproduce 2, 3, 5, 7, and 11.

The fields $\mathbf{Q}(\exp(2\pi i/87))$ and $\mathbf{Q}(\exp(2\pi i/77))$ occurring as the last two entries of Table 3.1 have class numbers $1536 = 2^9 3$ and $1280 = 2^8 5$. The corresponding Hilbert class fields, of degree $56 \cdot 1536 = 86016$ and $60 \cdot 1280 = 76800$, account for the two highest points on Figure 4.1.

For every $d < \Omega$, there is an upper bound $u(d)$ on the degree of a number field satisfying the generalized Riemann hypothesis with root discriminant $\leq d$. The upper bound is drawn in Figure 4.1 from the data given in [18]. The large empty region beneath the curve on the right is supportive of our expectation that the entire set $\mathcal{K}(\Omega)$ is finite.

The previous section can be viewed as the first step towards finding $\mathcal{K}(G,\Omega)$ for all solvable G, by induction on the solvable length of G. In this light, the current section represents a relatively easy part of the second step. To pursue the

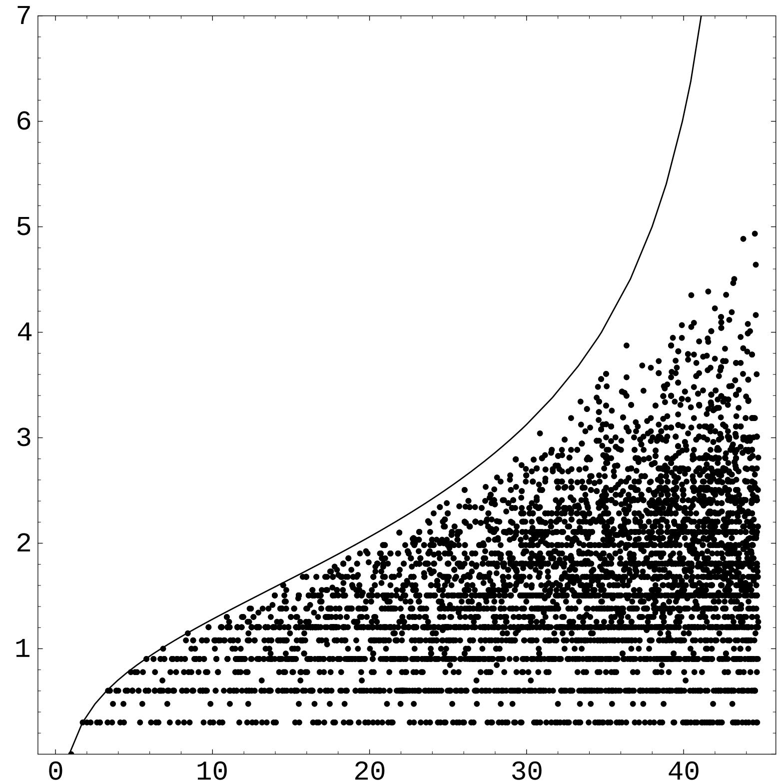


FIGURE 4.1. The ordered pairs $(d(H), \log_{10}[H : \mathbf{Q}])$ for H the class field of K, and K running over the 7063 abelian fields with $d(K) < \Omega$. Different K can have the same H and different H can give rise to the same point. Altogether, there are 3954 distinct ordered pairs plotted on this figure.

second step completely, one would have to make a careful analysis of ray class fields. As an indication of how many more fields we would find if we took the second step completely, consider the semidirect product groups $S_3 = C_3 : C_2$, $A_4 = V : C_3$, $D_5 = C_5 : C_2$, and $F_5 = C_5 : C_4$. In each case, a field K with Galois group $G = G_2 : G_1$ is seen by the techniques of this section if and only if the relative extension K/K_1 corresponding to G_2 is unramified. In this section, we see respectively 217, 2, 118, and 38 fields in these $\mathcal{K}(G, \Omega)$. From Theorem 5.1 of the next section, we know that there are respectively 393, 57, 28, and 64 more fields in $\mathcal{K}(G, \Omega)$.

5. DEGREES 3-6

This section centers on the following theorem.

Theorem 5.1. *One has*

$$\begin{array}{llll} & & & |\mathcal{K}(S_3,\Omega)| = 610, \\ & |\mathcal{K}(D_4,\Omega)| = 1425, & |\mathcal{K}(A_4,\Omega)| = 59, & |\mathcal{K}(S_4,\Omega)| = 527, \\ |\mathcal{K}(D_5,\Omega)| = 146, & |\mathcal{K}(F_5,\Omega)| = 102, & |\mathcal{K}(A_5,\Omega)| = 78, & |\mathcal{K}(S_5,\Omega)| = 192, \\ |\mathcal{K}(C_3^2.C_4,\Omega)| \geq 17, & |\mathcal{K}(C_3^2.D_4,\Omega)| \geq 137, & |\mathcal{K}(A_6,\Omega)| = 5, & |\mathcal{K}(S_6,\Omega)| = 13. \end{array}$$

If a degree n field has Galois root discriminant d, then its absolute discriminant is at most d^n. For $n = 3$, 4, this means that the fields we seek are all splitting fields of polynomials in the tables of [2]. For $n = 5$ and especially $n = 6$, the tables do not go nearly far enough. We ran computer searches for degree n fields whose Galois closures are the fields we want. Some details of these searches are given in the case $n = 6$ at the end of this section.

Our searches are guaranteed to find a degree n field only when the field is primitive, i.e. has no subfield besides $\mathbf{Q}$ and itself. For the prime degree $n = 5$, this does not pose a problem, and our search found all sub-Ω fields with the five possible Galois groups C_5, D_5, F_5, A_5, and S_5.

TABLE 5.1. The field with the minimal Galois root discriminant d_G for some small groups G.

G	Defining polynomial	Slope data		d_G
S_3	$x^3 - x^2 + 1$	23_2		4.80
D_4	$x^4 - x^3 - 3x^2 - x + 1$	3_3	7_2	6.03
A_4	$x^4 - 2x^3 + 2x^2 + 2$	$2[2,2]$	7_3	10.35
S_4	$x^4 - 2x^3 - 4x^2 - 6x - 2$	$2[\frac{4}{3},\frac{4}{3}]$	11_4	13.56
D_5	$x^5 - 2x^4 + 2x^3 - x^2 + 1$	47_2		6.85
F_5	$x^5 - 2$	2_5	$5[\frac{5}{4}]$	11.08
A_5	$x^5 - x^4 + 2x^2 - 2x + 2$	$2[2,2]$	17_3	18.70
S_5	$x^5 - 2x^4 + 4x^3 - 4x^2 + 2x - 4$	$2[\frac{8}{3},\frac{8}{3}]$	3_5 5_2	21.54
$C_3^2 : C_4$	$x^6 + 6x^4 - 3x^3 + 9x^2 - 9x + 1$	$3[2,2]$	5_4	≤ 23.57
$C_3^2 : D_4$	$x^6 + x^4 - 2x^3 + 3x^2 - x + 1$	$3[\frac{3}{2}]$	11_4	≤ 21.76

For the composite degree $n = 6$, there are twelve possible imprimitive Galois groups and four possible primitive Galois groups, with all imprimitive groups being solvable and all primitive groups non-solvable. Among the twelve imprimitive Galois groups, five have been seen before, $T1 \cong C_6$, $T2 \cong S_3$, $T4 \cong A_4$, and $T7 \cong T8 \cong S_4$. Five are product groups, $T3 \cong S_3 \times C_2$, $T5 \cong S_3 \times C_3$, $T6 \cong A_4 \times C_2$, $T9 \cong S_3 \times S_3$, and $T11 \cong S_4 \times C_2$. In these product cases, $\mathcal{K}(Q_1 \times Q_2, \Omega)$ can be obtained by composing fields in $\mathcal{K}(Q_1, \Omega)$ and $\mathcal{K}(Q_2, \Omega)$ and selecting out those fields which are sub-Ω, in the spirit of Section 12. The most interesting imprimitive cases are $T10 \cong C_3^2 : C_4$ and $T13 \cong C_3^2 : D_4$. Here we expect that the list of fields our search found is complete but there is no guarantee. To rigorously obtain the complete list, one could carry out class field theory computations as in [10, §3.4] or implement a targeted version of the relative searches first introduced in [16]. The primitive groups are $T12 \cong A_5$, $T14 \cong S_5$, $T15 = A_6$ and $T16 = S_6$. The groups

TABLE 5.2. The five fields in $\mathcal{K}(A_6, \Omega)$. For each, a twin pair of defining polynomials is given.

#	GRD	Polynomials	Slope data
1	31.66	$x^6 - 3x^5 + 3x^4 - 6x^2 + 6x - 2$	$2\left[\frac{8}{3}, \frac{8}{3}\right]$ $3[2,2]$
		$x^6 - 3x^4 - 12x^3 - 9x^2 + 1$	
2	37.23	$x^6 - 6x^3 - 6x^2 - 6x - 2$	$2\left[\frac{4}{3}, \frac{4}{3}\right]$ $3\left[\frac{3}{2}, \frac{3}{2}\right]$ 13_2
		$x^6 - 3x^5 + 3x^4 - 2$	
3	41.17	$x^6 - 3x^5 + 6x^4 + 17x^3 - 57x^2 + 69x - 47$	$3\left[\frac{3}{2}, \frac{5}{2}\right]$ 7_4
		$x^6 - 3x^5 + 3x^4 + 9x^3 - 18x^2 - 9x + 18$	
4	43.41	$x^6 - 3x^3 - 3x + 4$	$3\left[\frac{3}{2}, \frac{3}{2}\right]$ 29_3
		$x^6 - 3x^5 - 3x^4 + 11x^3 + 6x^2 + 75x + 50$	
5	44.67	$x^6 - x^5 + 2x^4 - 3x^2 + 2x - 4$	7_3 149_2
		$x^6 - 6x^4 - 7x^3 + 19x^2 + 7x - 15$	

A_5 and S_5 were already treated by the quintic search, so the main purpose of the sextic search is to find the A_6 and S_6 fields.

For the groups besides A_6 and S_6, information on the field with the smallest root discriminant is given in Table 5.1. The corresponding list of sub-Ω fields is given at [9]. For A_6 and S_6 we give these lists here. For each field K, we give two polynomials $f_a(x)$ and $f_b(x)$ defining non-isomorphic root fields $\mathbf{Q}[x]/f_a(x) \not\cong \mathbf{Q}[x]/f_b(x)$ embeddable in the common splitting field K. Similar twinning phenomena appear again for $PSL_2(7)$ and $PSL_2(11)$ in Sections 7 and 11 respectively.

Our sextic search consisted of a great many individual cases, with each case consisting of one ramification pattern for each ramifying prime. Each individual case was a Hunter-type search, as described in [5, §9.3], with the search region reduced by p-adic conditions, as described in [11]. The possible contribution of p to a Galois root discriminant of a sextic field is p^{β_p} where

$$\beta_2 \in \left\{0, \tfrac{2}{3}, \tfrac{4}{5}, 1, \tfrac{7}{6}, \tfrac{4}{3}, \tfrac{3}{2}, \tfrac{19}{12}, \tfrac{7}{4}, \tfrac{11}{6}, 2, \tfrac{25}{12}, \tfrac{13}{6}, \tfrac{9}{4}, \tfrac{7}{3}, \tfrac{31}{12}, \tfrac{11}{4}, 3\right\},$$
$$\beta_3 \in \left\{0, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{7}{6}, \tfrac{43}{36}, \tfrac{4}{3} \tfrac{25}{18}, \tfrac{3}{2}, \tfrac{31}{18}, \tfrac{16}{9}, \tfrac{11}{6}, \tfrac{37}{18}, \tfrac{25}{12}, \tfrac{13}{6}\right\},$$
$$\beta_5 \in \left\{0, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{5}{6}, \tfrac{23}{20}, \tfrac{13}{10}, \tfrac{31}{20}, \tfrac{8}{5}, \tfrac{39}{20}\right\},$$

and otherwise $\beta_p \in \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\right\}$. This yields 8154 possible GRDs in the interval $(1, \Omega)$, with 482, 3424, 3874, and 374 GRDs corresponding to 1, 2, 3, and 4 ramifying primes respectively.

A given possible GRD typically corresponds to several searches. Consider for example the common case of $\beta_p = 1/2$. This means that $p \geq 3$ and the p-adic ramification structure is either 21111, 2211, or 222 corresponding to p, p^2, or p^3 exactly dividing the discriminant of the sextic field sought. In this case, p would contribute a factor of 3 to the number of searches associated to the given GRD. The searches belonging to a given GRD can vary substantially in length. For example, for the $2003^{1/2}$ search, the 21111 search took approximately 1 second and looked at 18 polynomals. The 2211 search took approximately 22 minutes and looked at 2200 polynomials. We did not have to do the 222 search because sextic twinning takes fields with discriminant $\pm p$ into fields of $\pm p^3$.

TABLE 5.3. The thirteen fields in $\mathcal{K}(S_6, \Omega)$.

#	GRD	Polynomials	Slope data		
1	33.50	$x^6 - 2x^5 - 4x^4 + 12x^3 - 14x^2 + 8x - 4$ $x^6 - 2x^5 - x^4 + 6x^3 - 2x^2 - 4x - 1$	$2[2, 2, 3]$	3_5	5_3
2	37.53	$x^6 - 3x^5 + 5x^3 - 5$ $x^6 - 3x^5 + 5x^3 + 3x + 1$	2_3	$3\left[\frac{5}{4}, \frac{5}{4}\right]$	$5\left[\frac{5}{4}\right]$
3	38.15	$x^6 - 2x^5 + 5x^4 - 5x^2 + 10x - 5$ $x^6 - 2x^5 + 5x^4 - 10x^2 + 8x - 6$	$2\left[\frac{8}{3}, \frac{8}{3}, 3\right]$	$5\left[\frac{5}{4}\right]$	
4	38.40	$x^6 + 3x^4 - 4x^3 + 9x^2 - 6x + 1$ $x^6 + 3x^4 - 8x^3 - 9x^2 - 12x + 1$	2_5	$3\left[\frac{9}{4}, \frac{9}{4}\right]$	5_2
5	39.08	$x^6 - 2x^5 + 4x^4 - 8x^3 + 2x^2 + 24x - 20$ $x^6 - 4x^4 - 2x^3 + x^2 - 2x - 5$	$2[2, 2, 3]$	3_2	7_5
6	39.44	$x^6 - x^5 - 4x^4 + 6x^3 - 6x + 5$ $x^6 - x^5 + 7x^4 - 6x^3 + 11x^2 - 10x + 4$	11_2	41_3	
7	40.92	$x^6 - 3x^5 + 3x^4 + 8x^3 - 12x^2 + 12x - 4$ $x^6 + 2x^4 - 2x^3 - 3x^2 - 6x - 3$	$2[2, 2]$	$3[2]$	5_4
8	41.05	$x^6 - 9x^4 - 2x^3 + 9x^2 + 6x + 2$ $x^6 + 5x^4 - 2x^3 + 9x^2 - 8x + 16$	$2[2]$	$3[2]$	7_5
9	41.80	$x^6 - 6x^4 - 4x^3 + 18x^2 + 12x - 26$ $x^6 + 3x^4 - 4x^3 - 6x - 2$	$2\left[\frac{4}{3}, \frac{4}{3}, 3\right]$	$3\left[\frac{9}{4}, \frac{9}{4}\right]$	
10	42.25	$x^6 - 6x^4 - 4x^3 + 6x^2 - 6$ $x^6 - 6x^4 - 4x^3 + 6x^2 - 4$	$2\left[\frac{8}{3}, \frac{8}{3}, 3\right]$	$3[2, 2]$	
11	43.55	$x^6 - 2x^5 + 3x^4 + 6x^3 - 6x^2 + 15x + 15$ $x^6 - 2x^5 + 2x^4 - 2x^2 - 7x - 1$	3_5	5_2	23_3
12	44.26	$x^6 - 9x^4 - 16x^3 - 9x^2 - 6x - 1$ $x^6 - 3x^5 + 4x^3 - 4$	2_5	$3[2, 2]$	13_2
13	44.35	$x^6 - 2x^5 - 3x^4 + 6x^3 + 16x^2 - 32x + 8$ $x^6 - 2x^5 + 6x^4 - 8x^3 - 4x^2 + 8x - 4$	$2\left[\frac{4}{3}, \frac{4}{3}, 2\right]$	5_3	7_6

All together, the sextic searches here required several months of computer time. For comparison, consider the determination in [10] and [11] respectively of all primitive sextics and all septics ramified at 2 and 3 only. In each case, we had to search root discriminants well beyond Ω, in fact to $2^3 3^{13/6} \approx 86.47$ and $2^{37/12} 3^{13/6} \approx 91.61$. However these searches now take 2 hours and 13 hours respectively, as reported in more detail in [11].

6. Connections with automorphic forms

Some number fields are related to classical holomorphic forms on the complex upper half plane. We review this connection here using the A_6 number fields on Table 5.2 as examples. Our discussion in this section explains the meaning of the identities involving modular forms in the next four sections, as well as our entire approach to $SL_2(8)$ fields in Section 9. Other number fields are related to other sorts of automorphic forms and we also mention such connections here and in later sections.

We work inside the formal power series ring $\mathbf{C}[[q]]$. For each N, one has a finitely generated subring $M(N) = \bigoplus_k M_k(N)$ of modular forms on the group $\Gamma_1(N)$, with the weight k running over $0, 1/2, 1, \ldots$. Elements of $M(N)$ can be thought of as functions on the upper half z-plane via $q = e^{2\pi i z}$. Another conceptual point of view is that $M(N)$ is a projective coordinate ring for the modular curve $X_1(N)$. However neither of these viewpoints enters into the rest of the discussion, and it will suffice to think of modular forms simply as power series.

For k integral, one has the space of cuspidal newforms $S_k^{\mathrm{new}}(N)$ inside $M_k(N)$. This space is a direct sum of subspaces $S_k^{\mathrm{new}}(N, \chi)$ indexed by Dirichlet characters $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ satisfying $\chi(-1) = (-1)^k$. The only characters which will appear explicitly in this paper are the trivial character χ_1 and the quadratic Jacobi symbol characters $\chi_D(\cdot) = \left(\frac{D}{\cdot}\right)$ for D a discriminant of a quadratic field.

For p a prime not dividing N, one has the Hecke operator T_p on $S_k^{\mathrm{new}}(N, \chi)$ given by $T_p(\sum a_n q^n) = \sum (a_{pn} + \chi(p) p^{k-1} a_{n/p}) q^n$, with $a_{n/p}$ understood to be 0 if n/p is not integral. The T_p are semisimple commuting operators and all their simultaneous eigenspaces have dimension one. Say an eigenform is normalized if $a_1 = 1$. The set $S_k^{\mathrm{prim}}(N, \chi)$ of normalized eigenforms is then a basis for $S_k^{\mathrm{new}}(N, \chi)$. These primitive forms are tabulated for a broad range of (k, N, χ) at [24].

Let $\overline{\mathbf{Z}} \subset \mathbf{C}$ be the ring of algebraic integers. Any primitive form f lies in $\overline{\mathbf{Z}}[[q]]$. For a prime ℓ, fix an ideal in $\overline{\mathbf{Z}}$ with residue field $\overline{\mathbf{F}}_\ell$, an algebraic closure of $\mathbf{F}_\ell$. Via this choice, a primitive form gives an element of $\overline{\mathbf{F}}_\ell[[q]]$.

A representation $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\overline{\mathbf{F}}_\ell)$ is said to correspond to a primitive form $f \in \overline{\mathbf{F}}_\ell[[q]]$ if and only if it is unramified at primes not dividing $N\ell$ and for all such primes the Frobenius class $\rho(\mathrm{Fr}_p)$ has characteristic polynomial $x^2 - a_p x + \chi(p) p^{k-1}$. Every f has a corresponding ρ. Henceforth we restrict attention to ρ which are semisimple, i.e. either irreducible or the the sum of two irreducibles. Then ρ is unique up to conjugation. Also ρ is odd in the sense that ρ(complex conjugation) has eigenvalues -1 and 1. Much of the ramification of ρ can be given directly in terms of f.

Conversely, suppose we are given an odd semisimple $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\overline{\mathbf{F}}_\ell)$. Then Serre's conjecture says that it should come from a modular form via the above construction. From the ramification in ρ, one can even specify the data (k, N, χ). In fact, the examples we give in this paper illustrate many aspects of the recipe in [23].

Consider now Galois fields $K \subset \mathbf{C}$ with $\mathrm{Gal}(K/\mathbf{Q})$ of the form $PGL_2(\lambda)$ for $\lambda = \ell^f$ a prime power. If $\ell \neq 2$ we allow also the index two subgroup $PSL_2(\lambda)$ but suppose further that K is not totally real. Then, since any homomorphism $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to PGL_2(\overline{\mathbf{F}}_\ell)$ lifts to a representation $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\overline{\mathbf{F}}_\ell)$, all such K are expected to come from primitive forms.

Explicitly, such a K can be given as the splitting field of a degree $\lambda+1$ polynomial $f(x) \in \mathbf{Z}[x]$. For p not dividing the discriminant of K, information on the Frobenius element Fr_p can be obtained by the partition μ_p of $\lambda+1$ whose parts give the degrees of the irreducible factors of $f(x) \in \mathbf{Z}_p[x]$. The possible partitions are $(u, \ldots, u, 1, 1)$ for any factorization $\lambda - 1 = um$, $(u, \ldots, u)$ for any factorization $\lambda + 1 = um$, and also $(\ell, \ldots, \ell, 1)$. The order of the Frobenius element is the least common multiple of the parts of μ_p, i.e. just u, u, and ℓ in the three cases. The field K comes from a primitive form $f = \sum a_n q^n$ if and only if

(6.1) the order of the Frobenius element $\mathrm{Fr}_p \overset{\star}{=}$

$$\text{the order of} \begin{pmatrix} 0 & -1 \\ \chi(p)p^{k-1} & a_p \end{pmatrix} \text{ in } PGL_2(\lambda)$$

for p not dividing $N\ell$. The right side depends only on the quotient $a_p^2/\chi(p)p^{k-1}$ in $\mathbf{F}_\lambda \subset \overline{\mathbf{F}}_\ell$. Also $u \overset{\star}{=} v$ means $u = v$ or $(u,v) = (1,\ell)$. In the cases $PSL_2(7) \cong GL_3(2)$, $PSL_2(9) \cong A_6$ and $PSL_2(11)$, we will work with polynomials of degree 7, 6, and 11 rather than 8, 10, and 12 respectively.

One way to construct modular forms is by θ-series. For t a positive integer, let

$$\theta_t \quad = \quad \sum_{k=-\infty}^{\infty} q^{k^2 t} = 1 + 2q^t + 2q^{4t} + 2q^{9t} + \cdots .$$

The form θ_t lies in $M_{1/2}(4t)$. As an example of the usefulness of theta series, one has $M(8) = \mathbf{C}[\theta_1, \theta_2]$. In general, it is useful to introduce the abbreviations

$$\begin{aligned} \hat{\theta}_t &= \frac{1}{2}\theta_t - \frac{1}{2}\theta_{4t}, \\ \check{\theta}_t &= 2\theta_t - \theta_{t/4} \end{aligned}$$

to keep formulas relatively simple. Here the latter abbreviation is used only when t is a multiple of 4.

Let f_{6a} and f_{6b} denote the first two polynomials in Table 5.2 with common splitting field $K_{A_6,1}$. A primitive modular form giving rise to $K_{A_6,1}$ is

$$\begin{aligned} f_{A_6,1} \quad = \quad & \hat{\theta}_1\theta_4\theta_8\check{\theta}_{32}\left[(-1-\sqrt{2})\theta_4^2 + (2+\sqrt{2})\theta_8^2\right] + \\ & \qquad i\hat{\theta}_1\hat{\theta}_2\check{\theta}_{16}\check{\theta}_{32}\left[2\theta_4^2 + 2\sqrt{2}\theta_8^2\right] \in S_3^{\mathrm{prim}}(128, \chi_{-4}). \end{aligned}$$

Table 6.1 illustrates the correlation between the factorization patterns $\mu_{6a,p}$ and $\mu_{6b,p}$ of f_{6a} and f_{6b} over $\mathbf{Z}_p$ on the one hand, and the quantities $d_p = a_p^2/\chi_{-4}(p)p^2$ in $\mathbf{F}_9 = \mathbf{F}_3[i]$ on the other.

TABLE 6.1. Behavior of primes $5 \leq p \leq 97$ in the first A_6 field $K_{A_6,1}$ and the matching modular form $f_{A_6,1}$.

Class	$\mu_{6a,p}$	$\mu_{6b,p}$	d_p	Primes
$1A$	1^6	1^6	1	
$2A$	2211	2211	0	67
$3A$	3111	33	1	
$3B$	33	3111	1	$11, 73, 79$
$4A$	42	42	2	$13, 19, 31, 47, 71, 89$
$5A$	51	51	i	$5, 7, 37, 41, 43, 53, 83$
$5B$	51	51	$-i$	$17, 23, 29, 59, 61, 97$

To see definitively that $K_{A_6,1}$ really comes from $f_{A_6,1}$ one can proceed as follows. Certainly $f_{A_6,1}$ gives rise to an A_6 field K ramified only at 2 and 3. However there are only four such fields [10], and the three fields different from $K_{A_6,1}$ do not match the Frobenius data in Table 6.1. Therefore $K = K_{A_6,1}$. We have used complete tables to obtain similar definitive matching for many of the $A_5 \cong SL_2(4) \cong PSL_2(5)$ and $S_5 \cong PGL_2(5)$ fields found by the search of the previous section. Similarly, for the next three A_6 fields one has definitive matches $f_{A_6,2} \in S_2^{\text{prim}}(104, \chi_{13})$, $f_{A_6,3} \in S_4^{\text{prim}}(49, \chi_1)$, and $f_{A_6,4} \in S_2^{\text{prim}}(29, \chi_1)$. However, even in this small group setting, the desired primitive form may be beyond the tables in [24] when ℓ does not ramify in K. Thus we do not presently have a matching form for the fifth A_6 field. If one allows also purely characteristic ℓ modular forms, then these unramified-at-ℓ fields should match modular forms of weight 1. Sections 7, 8, 10, and 11 provide several examples of fields for which the corresponding characteristic ℓ weight one form has not been found.

The matches in the next four sections require not only the theta series we have introduced, but also the twisted theta series

$$\phi_t = \sum_{n=-\infty}^{\infty} \chi_8(n) q^{tn^2} \in M_{1/2}(256t). \tag{6.2}$$

Also we use the eta-function

$$\eta_t = q^{t/24} \prod_{k=1}^{\infty} (1 - q^{tk}). \tag{6.3}$$

If t is a multiple of 24 then $\eta_t \in M_{1/2}(24t)$.

Some of the matches in the next sections are only numeric in the sense that we have checked agreement in (6.1) for $p \leq 1000$ but we haven't established full agreement rigorously. To definitively establish these matches, one approach would be to try to use an effective Chebotarev density theorem. Of course, theoretical results towards Serre's conjecture might immediately settle the issue. There is such a result in the case of $PSL_2(9)$ and $PGL_2(9)$ [8]. It applies to our second through fourth A_6 fields, since its hypothesis on ramification at 3 is satisfied. It doesn't apply to our first or fifth A_6 field, nor to (10.1).

To find automorphic matches for the S_6 fields of the previous section, one needs to go beyond classical modular forms. When the quadratic subfield F is real, the setting of Hilbert modular forms over F should provide matches. For example, $K_{S_6,j}$ contains $F = \mathbf{Q}(\sqrt{5})$ for $j = 2, 4, 7$, and 11. These four fields have been seen numerically by Diamond and Dembele in their computations with Hilbert modular forms over $\mathbf{Q}(\sqrt{5})$. Another place that should provide matches for S_6 fields is Siegel modular forms of genus two, via $S_6 \cong Sp_4(2)$.

7. Degree 7

There are three non-solvable septic groups, $SL_3(2)$, A_7, and S_7. Run times would be too long to fully compute $\mathcal{K}(G, \Omega)$ for all three groups. However early on we found

$$f_{A_7}(x) = x^7 - 3x^6 + 3x^5 + 3x^4 - 9x^3 + 3x^2 + x - 3 \tag{7.1}$$

with Galois group A_7, slope data $(2_7,\ 3[3/2, 3/2],\ 7_5)$, and Galois root discriminant $2^{6/7}3^{25/18}7^{4/5} \approx 39.516$. We carried out a search for alternating septics only and cutoff 39.52, obtaining the following result:

Theorem 7.1. *One has* $|\mathcal{K}(SL_3(2), 39.52)| = 8$ *and* $|\mathcal{K}(A_7, 39.52)| = 1$.

TABLE 7.1. The eight fields in $\mathcal{K}(SL_3(2), 39.52)$ and then nine more fields in $\mathcal{K}(SL_3(2), \Omega)$.

#	GRD	Polynomials	Slope data
1	32.25	$x^7 - x^6 - 9x^5 - x^4 + 19x^3 + 21x^2 - 23x - 13$ $x^7 - x^6 - 9x^5 - x^4 + 19x^3 + 21x^2 + 21x + 9$	2_7 3_4 11_7
2	32.25	$x^7 - 2x^6 + 2x^4 - 2x^3 + 2x^2 - 2$ $x^7 - 3x^6 + 3x^5 - x^4 - 5x^3 + 5x^2 + 3x - 1$	2_7 317_2
3	35.06	$x^7 - x^6 + 2x^5 - 12x^4 - 14x^3 + 10x^2 + 10x - 2$ $x^7 - 3x^6 + 7x^5 - 5x^4 - 12x^3 + 32x^2 - 36x + 4$	$2\left[\frac{8}{3}, \frac{8}{3}\right]$ 11_7
4	37.64	$x^7 - x^6 - 3x^5 + x^4 + 4x^3 - x^2 - x + 1$ $x^7 - 2x^6 - x^5 + 4x^4 - 3x^2 - x + 1$	13_2 109_2
5	38.13	$x^7 - 3x^6 + 3x^5 + x^4 - 3x^3 + x^2 - x - 1$ $x^7 - 3x^6 + x^5 + 3x^4 - x^3 + x^2 - 3x - 1$	2_7 443_2
6	38.72	$x^7 - 2x^6 + 2x^5 + 2x^4 - 4x^3 + 4x^2 - 4$ $x^7 - 2x^6 - 2x^5 + 6x^4 - 4x^3 - 2x^2 + 4x - 2$	2_7 457_2
7	39.16	$x^7 - x^6 + x^5 + x^4 - 3x^3 + 5x^2 - 2x - 1$ $x^7 - x^6 + x^5 - 6x^4 + 4x^3 + 5x^2 - 2x - 1$	7_4 19_4
8	39.20	$x^7 - x^6 - 9x^5 - x^4 + 8x^3 - 12x^2 - 12x - 2$ $x^7 - 2x^6 + 8x^5 - 8x^4 - 4x^3 + 12x^2 - 20x + 8$	$2\left[\frac{4}{3}, \frac{4}{3}\right]$ 5_2 11_7
	39.54	$x^7 - x^6 + 3x^5 - 5x^4 + 3x^3 + 9x^2 - 7x + 1$ $x^7 - 2x^6 - 4x^5 + 6x^4 + 8x^3 - 22x^2 + 16x - 2$	2_7 61_4
	39.55	$x^7 - x^6 + 13x^5 - 23x^4 + 8x^3 - 23x^2 + 21x + 9$ $x^7 - x^6 - 9x^5 + 21x^4 - 3x^3 - 23x^2 + 10x + 9$	3_2 5_3 11_7
	40.08	$x^7 - 7x - 3$ $x^7 - 7x^4 - 21x^3 + 21x^2 + 42x - 9$	$3\left[\frac{3}{2}\right]$ $7\left[\frac{4}{3}\right]$
	41.35	$x^7 - 2x^5 - 4x^4 - 2x^3 - 2x^2 + 2$ $x^7 - 3x^6 + 5x^5 - 7x^4 + 5x^3 - 3x^2 - x + 1$	2_7 521_2
	43.26	$x^7 - 14x^4 - 21x^3 - 42x^2 - 28x + 30$ $x^7 - 7x^5 + 21x^3 - 14x^2 - 7x + 4$	$2\left[\frac{4}{3}, \frac{4}{3}\right]$ 3_2 $7\left[\frac{4}{3}\right]$
	43.40	$x^7 + 2x^5 - 12x^3 - 2x^2 + 6x - 2$ $x^7 - 2x^6 - 2x^5 + 2x^4 - 8x^2 + 4x + 4$	2_7 7_4 31_2
	44.10	$x^7 - 3x^6 + 3x^5 + 3x^4 - 18x^3 + 28x^2 - 24x + 8$ $x^7 - 7x^5 - 10x^4 + 3x^3 + 3x + 2$	$2\left[\frac{4}{3}, \frac{4}{3}\right]$ 53_4
	44.27	$x^7 - 2x^6 - 4x^5 + 10x^3 + 4x^2 - 10x - 6$ $x^7 - 2x^6 - 6x^5 + 14x^4 - 70x^2 - 42x + 18$	2_7 3_2 53_3
	44.50	$x^7 - 7x^5 - 14x^4 - 7x^3 - 7x + 2$ $x^7 - 14x^3 - 14x^2 + 7x + 22$	$2[2, 3]$ $7\left[\frac{4}{3}\right]$

One reason for restricting attention to alternating septics is simply to cut down on the number of cases. Another is that a particularly difficult p-adic ramification partition is 2221. In the setting of sextics, we twinned away from 222 when it occurred for a problematically large prime. Here twinning is not available, but when one restricts to alternating septics, the structure 2221 simply does not occur.

All together, the septic search behind Theorem 7.1 took several months, just like the sextic search discussed at the end of Section 5.

From partial searches we know further that

$$|\mathcal{K}(SL_3(2),\Omega)| \geq 17,$$
$$|\mathcal{K}(S_7,\Omega)| \geq 1.$$

The S_7 field is defined by

$$f_{S_7}(x) = x^7 - 3x^6 + 6x^5 - 5x^4 + 6x^2 - 8x + 6, \tag{7.2}$$

with slope data $(2[2,2],\ 3[3/2],\ 5_7)$ yielding root discriminant $2^{3/2}3^{7/6}5^{6/7} \approx 40.49$. The seventeen $SL_3(2)$ fields are given on Table 7.1. The first two fields on this $SL_3(2)$ table are properly ordered, as the first has GRD $d_{SL_3(2)} = 2^{6/7}3^{3/4}11^{6/7} \approx 32.247$ while the second has GRD $2^{6/7}317^{1/2} \approx 32.252$.

The eleventh listed field on the $SL_3(2)$ table is the famous Trinks field. A modular form numerically matching the Trinks field via $SL_3(2) \cong PSL_2(7)$ was given by Mestre in [23, §5.5]. We find that a modular form numerically matching the last field in Table 7.1 is

$$f_{44.50} = \phi_1\theta_8\left(\frac{1}{2}\check{\theta}_{32}^2\theta_8^2 - \sqrt{-2}\hat{\theta}_2\check{\theta}_{32}^3 + 2\hat{\theta}_2^2\theta_8^2 + 4\sqrt{-2}\hat{\theta}_2^3\check{\theta}_{32}\right) \in S_3^{\mathrm{prim}}(256,\chi_{-4}).$$

The fields $K_{SL_3(2),j}$ for $j = 2,\ 5,\ 6$, are numerically matched with automorphic forms on GL_3 in [1].

8. Degree 8

There are four non-solvable groups in degree eight of the form $H^m.A$ discussed in the introduction, $PSL_2(7)$, $PGL_2(7)$, A_8, and S_8. The group $PSL_2(7)$ was already treated in the previous section via $PSL_2(7) \cong SL_3(2)$. Here we report on twenty sub-Ω fields with $G = PGL_2(7)$. By extrapolation from lower degrees, and also from partial octic searches, we expect that there are no sub-Ω fields with $G = A_8$ or S_8.

Thirteen of the twenty known fields in $\mathcal{K}(PGL_2(7),\Omega)$ were found by specializing three point covers. We worked with four covers with numerical invariants as in

Table 8.1. Information corresponding to four octic three point covers.

N	λ_0	λ_1	λ_∞	g	$d_N(t)$	Bad p		
$8a$	3311	2222	71	0	-7	2	3	7
$8b$	611	22211	71	0	$-7t(t-1)$	2	3	7
$8c$	44	22211	611	0	$-7(t-1)$	2	3	7
$8d$	44	2222	71	1	-7	2		7

Table 8.1 and equations as follows:

$$f_{8a}(t,x) = (x^2+5x+1)^3(x^2+13x+49) - 2^6 3^3 tx,$$

$$f_{8b}(t,x) = x^6(x^2-x+7) - 2^2 3^3 t(x-1),$$

$$f_{8c}(t,x) = 3^3(x^2+7)^4 - 2^{10}t(7x^2-6x+63),$$

$$\begin{aligned} f_{8d}(t,x) = (1-t)(x^2+14x+21)^4 + \\ (x^4+28x^3+238x^2+588x-455)^2 t - 2^{16}(t-1)tx. \end{aligned}$$

In cases $8a$, $8b$, and $8c$, as well as most cases in the sequel, the cover is a map from a projective line $\mathbf{P}^1_x$ to a projective line $\mathbf{P}^1_t$, ramified only over $t=0$, $t=1$, and $t=\infty$. Case $8d$ is more complicated as the covering curve has genus one. Always we indicate the corresponding ramification partitions by λ_0, λ_1, and λ_∞. The fact that the cover is ramified only above 0, 1, and ∞ is reflected in polynomial discriminants, e.g.,

$$D_{8a}(t) = -2^{48}3^{24}7^7 t^4(t-1)^4.$$

Specializing a cover means essentially plugging in a number $\tau \in \mathbf{Q} - \{0,1\}$ for t, and, to get a Galois field, taking the splitting field $K_{N,\tau}$ of $f_N(\tau,x)$ in $\mathbf{C}$. The contribution from p to the root discriminant of $K_{N,\tau}$ is a p-adically continuous function of τ. This function is complicated reflecting wild ramification for a finite set of bad primes. It is simple at the remaining primes, where the ramification can be at worst tame. The final column to explain in Table 8.1 is headed by $d_N(t)$. This gives the discriminant $D_N(t)$ modulo squares in $\mathbf{Q}(t)^\times$. So Covers $8a$ and $8d$ are only capable of yielding fields containing $\mathbf{Q}(\sqrt{-7})$ while Covers $8b$ and $8c$ can yield $PGL_2(7)$ fields containing any quadratic field and also $PSL_2(7)$ fields. See Section 13 for two examples treated in some detail, and also [21] for a more systematic presentation of the general technique of constructing number fields by specializing three point covers.

Cover $8a$, $8b$, $8c$, and $8d$ yielded respectively 9, 1, 0 and 7 fields. Some fields were repeated, giving thirteen distinct fields in all. We also did a naive search for fields given by polynomials with small coefficients. This yielded some of the same thirteen fields, and five more fields; these five extra fields were already in [14]. Finally a search tailored to find fields with discriminant of the form 2^a3^b found two more fields.

Tables 8.2 and 8.3 indicate a connection with elliptic curves and in the next four paragraphs we explain this connection. Cover $8a$ can be identified with the modular curve $X_0(7)$ covering the j-line $X_0(1)$. A cubic base-change of this cover is $X_0(14) \to X_0(2)$. This last map is equivariant with respect to the Atkin-Lehner operator w_2. Cover $8d$ is the corresponding quotient $X_0(14)/w_2 \to X_0(2)/w_2$. The

TABLE 8.2. The first ten known fields in $\mathcal{K}(PGL_2(7), \Omega)$.

GRD	Polynomial	Sources	Elliptic	Slope data
27.35	$x^8 - x^7 + 3x^6 - 3x^5 + 2x^4$ $-2x^3 + 5x^2 + 5x + 1$			53_6
30.46	$x^8 - 4x^6 - 8x^5 + 32x^3$ $+16x^2 - 1$	a $-7^5/2^1 3^8$	18816C1	$2[\frac{8}{3}, \frac{8}{3}]$ 3_7 7_2
31.49	$x^8 - 2x^7 + 7x^4 - 14x^2$ $+8x + 5$	a $2^4 23^3/3^3 5^7$ a $2^2 7^2/3^3$ d $1/2^6$	1960C1 392B1	$2[\frac{4}{3}, \frac{4}{3}]$ $7[\frac{3}{2}]$
31.60	$x^8 - 4x^7 + 21x^4 - 18x + 9$	a (6 j's) and d (5 t's); see (8.3)	24A1-6	$2[\frac{4}{3}, \frac{4}{3}]$ 3_7 7_8
31.64	$x^8 - 6x^4 - 48x^3 - 72x^2$ $-48x - 9$			$2[2, 3, \frac{7}{2}, \frac{9}{2}]$ 3_7
35.49	$x^8 - x^6 - 3x^5 - x^4 + 4x^3$ $+4x^2 - 2x - 1$			5_3 7_3 11_2
35.82	$x^8 - x^7 + 7x^6 + 7x^5 - 7x^4$ $+49x^3 - 35x^2 + 41x - 20$	a $-223^3/2^{14} 3^2$	2646B1	2_7 $3[\frac{3}{2}]$ 7_8
38.05	$x^8 - 2x^7 + x^6 + 4x^5 - x^4$ $+6x^3 + 3x^2 - 8x + 4$			$2[3]$ 181_2
39.62	$x^8 - 4x^7 + 14x^4 - 8x + 4$	a $2/3^3$ a $7^3/2^1 3^3$ d $-1/2^3$	128A1 128A2	$2[\frac{8}{3}, \frac{8}{3}]$ $7[\frac{7}{6}]$
39.67	$x^8 - 2x^7 + 14x^4 - 16x + 4$	a $-1/3^3$ a $31^3/2^3 3^3$ d $-7^2/2^5$	1568D1 1568D2	$2[2, 2]$ $7[\frac{3}{2}]$

situation is thus as follows:

$$
\begin{tikzcd}
\underline{8a} & & \underline{8d} \\
& X_0(14) \arrow[dl] \arrow[d] \arrow[dr] & \\
X_0(7) \arrow[d] & X_0(2) \arrow[dl] \arrow[dr] & X_0(14)/w_2 \arrow[d] \\
X_0(1) & & X_0(2)/w_2.
\end{tikzcd}
\tag{8.1}
$$

To discuss more than one cover at once, it is convenient to introduce new parameters. For a coordinate on $X_0(1)$ we replace t by j. On $X_0(2)$ we use a suitable coordinate u. On $X_0(2)/w_2$ we use our previous t. With these conventions, the two lower maps in (8.1) are given by

$$
\begin{aligned}
j &= \frac{(4u-1)^3}{27u}, \\
t &= \frac{-(u-1)^2}{4u}.
\end{aligned}
$$

TABLE 8.3. The remaining ten known fields in $\mathcal{K}(PGL_2(7),\Omega)$.

GRD	Polynomial	Sources	Elliptic	Slope data	
39.79	$x^8-x^7-4x^5+8x^3$ $+2x^2-2$			2_7 5_6	3_2 11_2
41.34	$x^8-x^7+x^6+4x^5-x^4$ $-3x^3+5x^2-2x+1$			1709_2	
42.43	$x^8-2x^7-3x^5+6x^4$ $+3x^3-3x^2+3$			3_7	29_6
42.59	$x^8-x^7-7x^6+7x^2$ $-27x-1$	a $2^2 7/3^3$ a $2^2 7^1 61^3/3^3$ d $7/2^4$; b $2^2 7^2/3^3$	196A1 196A2	2_3	$7\left[\frac{11}{6}\right]$
43.23	$x^8-14x^5-35x^4-42x^3$ $-14x^2-16x-14$	d $1/2^2$		$2[2,2]$ $7\left[\frac{7}{6}\right]$	3_2
43.65	$x^8-3x^7+7x^4-7x^2$ $-x-4$	a $-11^2/2^6 3^3$ a $-11^1 131^3/2^6 3^2$	121A2 121A1	$7\left[\frac{7}{6}\right]$	11_3
44.25	$x^8+x^7-7x^4+8x+1$			5_2 13_2	7_8
44.30	$x^8+2x^7-7x^4+4x+1$	a $2^5/3^3$ a $17^3/2^2 3^3$ d $5^2/2^4$	200B1 200B2	$2\left[\frac{4}{3},\frac{4}{3}\right]$ $7\left[\frac{7}{6}\right]$	5_2
44.45	$x^8-16x^5+42x^4-96x^3$ $+136x^2-48x-33$			$2\left[2,3,\frac{7}{2},\frac{9}{2}\right]$	$3\left[\frac{3}{2}\right]$
44.46	$x^8-2x^6-2x^5+6x^4$ $+8x^3+5x^2+x+1$			3_2	659_2

The equation

$$h(j,t) = 729j^2 - 54(512t^2 - 414t + 27)j + (16t+9)^3 = 0 \tag{8.2}$$

gives the curve $X_0(2)$ as a correspondence between the base projective line $\mathbf{P}^1_j$ of 8a and the base projective line $\mathbf{P}^1_t$ of 8d.

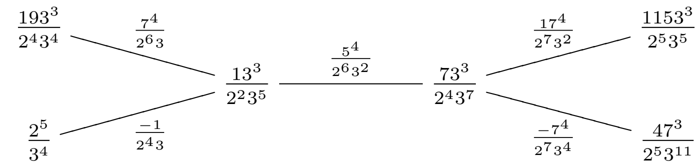
If a field K appears in Table 8.2 or 8.3 with "a j" listed as a source, then, for any elliptic curve E with J-invariant $1728j$, K embeds into the $GL_2(7)$ field L_E generated by the 7-torsion points of E. Because of quadratic twisting, L_E depends on the choice of E, but K doesn't. The table gives an elliptic curve with minimal conductor from [4] for every j.

For seven of the nine fields K coming from $8a$, more than one j gives rise to K. This multiplicity is explained by isogenies in all cases except the one involving the non-isogenous curves $1960C1$ and $392B1$. The isogenies are 2-isogenies except for $196A1 \overset{3}{\sim} 196A2$ and $121A1 \overset{11}{\sim} 121A2$.

A rational number $\tau \neq 0, 1$, considered as a point in $X_0(2)/w_2$, corresponds to an unordered pair of elliptic curves E_1 and E_2 with a 2-isogeny between them, with the J-invariants of E_1 and E_2 being 1728 times the roots of the polynomial $h(j,\tau)$ in (8.2). In cases 31.49, 42.59, and 43.23, the j-invariants of E_1 and E_2 are conjugate quadratic irrationalities. In cases 39.62, 39.67, and 44.30, the j-invariants of E_1 and

E_2 are both rational, and correspond to the previously discussed specializations of $8a$. The case 31.60 is similar but with a more complicated 2-isogeny tree:

(8.3)



Here, the vertices are labelled with j, and edges by t. Our cover $8b$ is similarly identified with $X_0(21)/w_3 \to X_0(3)/w_3$. The cover $8c$ is naturally identified with a map of Shimura curves associated with the quaternion algebra ramified at 2 and 3.

Some simple formulas for matching modular forms are as follows:

$$\begin{aligned}
f_{31.49} &= \eta_2^4\eta_4^4 && \in S_4^{\mathrm{prim}}(8,\chi_1),\\
f_{31.60} &= \eta_2\eta_4\eta_6\eta_{12} && \in S_2^{\mathrm{prim}}(24,\chi_1),\\
f_{39.62} &= \hat\theta_1\check\theta_8\check\theta_{16}\check\theta_{32} && \in S_2^{\mathrm{prim}}(128,\chi_1),\\
f_{39.67} &= \hat\theta_1\theta_2\theta_4\check\theta_8(\theta_2^2\check\theta_8^2+8\hat\theta_1^2\theta_4^2) && \in S_4^{\mathrm{prim}}(32,\chi_1),\\
f_{42.59} &= \eta_2^{12} && \in S_6^{\mathrm{prim}}(4,\chi_1).
\end{aligned}$$

These formulas are known rigorously through the connection with elliptic curves, which is direct in the cases of 31.60 and 39.62. Via $PGL_2(7) = SL_3(2).2$, the fields on Tables 8.2 and 8.3 containing an imaginary quadratic field F should also arise from holomorphic modular forms on $U_{2,1}$ over F.

9. Degree 9

Besides A_9 and S_9, there are two non-solvable degree nine groups, the simple group $T27 = SL_2(8)$ and its automorphism group $T32 = \Sigma L_2(8) = SL_2(8).3$. In this section, we present the fields we know in $\mathcal{K}(G,\Omega)$ for the latter two G and how we found them.

For $SL_2(8)$, [7] says that there are modular forms in characteristic two of weight 1 for the prime conductors $p = 1429$, 1567, 1613, 1693, and 1997 which conjecturally give rise to fields $K \subset \mathbf{C}$ with $\mathrm{Gal}(K/\mathbf{Q}) \cong SL_2(8)$ and root discriminant $\sqrt{p} < \Omega$. We wrote a specialized search program which inputs a prime p and outputs certain polynomials

$$f(x) = x^9 + \sum_{i=1}^{9} a_i x^{9-i}.$$

The program loops over small values of a_1, a_2, a_3, and a_4. It finds the integers a_5, a_6, a_7, a_8, and a_9 in $(-p/2, p/2)$ such that $f(x)$ factors modulo p into a linear polynomial times a quartic squared. It then outputs polynomials whose discriminant is a square. Looking among the polynomials output by the program, we found polynomials for each of the primes p.

Also for $SL_2(8)$, we searched the modular forms database [24] for conductors N in weight 2 which from the factorization modulo 2 of the Hecke polynomials on new forms should give rise to $SL_2(8)$ fields. We found that 10 appropriate forms, assuming generic behavior at 2, all coming from prime power conductors, namely $N = p^e = 97$, 109, 113, 127, 139, 149, 151, $169 = 13^2$, $243 = 3^5$, and $289 = 17^2$.

TABLE 9.1. The fifteen known fields in $\mathcal{K}(SL_2(8), \Omega)$. Each of them was found starting from a known modular form. R here is an abbreviation for $2[2, 2, 2]$.

GRD	Polynomial	Slope data
30.31	$x^9 - 3x^8 + 12x^6 - 14x^5 - 2x^4 + 12x^2 + x + 1$	R 13_7
32.18	$x^9 - 6x^6 + 18x^5 - 18x^4 + 36x^3 + 18x^2 + 27x + 22$	R $3\left[\frac{3}{2}, \frac{5}{2}\right]$
33.13	$x^9 - x^8 + 12x^6 - 12x^5 - 28x^4 + 48x^3 - 20x^2 + 5x - 1$	R 97_2
35.12	$x^9 - x^8 - 16x^6 + 36x^5 - 12x^4 - 8x^3 - 3x - 1$	R 109_2
35.76	$x^9 + 4x^7 - 10x^5 + 12x^4 - 12x^2 + 10x - 4$	R 113_2
37.80	$x^9 + x^7 - 4x^6 - 12x^4 - x^3 - 7x^2 - x - 1$	1429_2
37.91	$x^9 - x^8 + 8x^7 - 18x^6 + 28x^5 - 56x^4 + 68x^3 - 56x^2 + 26x - 6$	R 127_2
39.59	$x^9 - 2x^8 + 10x^7 - 25x^6 + 34x^5 - 40x^4 + 52x^3 - 45x^2 + 20x - 4$	1567_2
39.66	$x^9 + 8x^7 - 16x^6 + 30x^5 - 64x^4 + 80x^3 - 56x^2 + 22x - 4$	R 139_2
40.16	$x^9 - 2x^8 + 6x^6 - 8x^5 - 12x^4 + 12x^3 + 12x^2 - x + 1$	1613_2
41.06	$x^9 - 4x^8 + 10x^5 + 20x^4 - 72x^3 - 8x^2 + 161x - 128$	R 149_2
41.15	$x^9 - 18x^7 - 2x^6 + 85x^5 + 24x^4 - 8x^3 + 19x^2 + 5x - 14$	1693_2
41.33	$x^9 - 2x^8 + 2x^7 + 10x^6 - 18x^5 + 30x^4 + 66x^3 - 42x^2 + 71x + 152$	R 151_2
41.74	$x^9 + 34x^5 - 136x^4 + 272x^3 - 340x^2 + 238x - 68$	R 17_9
44.69	$x^9 - 3x^8 + 4x^6 + 5x^5 - 18x^4 + 23x^3 - 25x^2 + 16x - 4$	1997_2

We searched in the manner described in [11], simultaneously targeting behavior at 2 and p. Assuming generic behavior at 2, the decomposition group at 2 is the full group $T25 = C_2^3.C_7$ of upper triangular matrices in $SL_2(8)$. Certainly this implies that $f(x)$ factors 2-adically as an octic $g(x) \in \mathbf{Z}_2[x]$ times a linear factor. Moreover, from the complete enumeration of 2-adic octic fields [13], we know that there are only two possibilities for $L = \mathbf{Q}_2[x]/g(x)$, namely $L_1 = \mathbf{Q}_2[x]/g_1(x)$ and $L_2 = \mathbf{Q}_2[x]/g_2(x)$ with

$$\begin{aligned} g_1(x) &= x^8 + 2x^7 + 2x^4 + 2, \\ g_2(x) &= x^8 + 2x^7 + 2x^6 + 2. \end{aligned}$$

Local computations then say that for an octic Eisenstein polynomial $g(x) \in \mathbf{Z}_2[x]$, one has $\mathbf{Q}_2[x]/g(x) \cong L_j$ iff $g(x) \equiv g_j(x)$ (4). This statement formed the basis of our targeting at 2. The targeting at p, corresponding to tame ramification in every case except $p^e = 3^5$, was easier. We found a field for each of the ten N.

The modular form we started from to compute the second field first appeared in [15]:

$$\begin{aligned} (9.1) \qquad f_{32.18} &= (\alpha - 1 + T_2)(\eta_9^5\eta_{81}/\eta_3\eta_{27} + \eta_3\eta_{27}^5/\eta_9\eta_{27}) + \\ &\qquad (\alpha^2 - \alpha - 2)(\eta_3^2\eta_9\eta_{81} + 3\eta_3\eta_{27}\eta_{81}^2) \in S_2^{\mathrm{prim}}(243, \chi_1). \end{aligned}$$

For the others, we do not know of similarly explicit formulas.

Our fifteen fields in $\mathcal{K}(\Sigma L_2(8), \Omega)$ were found by specializing three point covers, with numeric invariants as in Table 9.2 and equations as follows:

TABLE 9.2. Invariants of three covers with generic Galois group $\Sigma L_2(8) = SL_2(8).3$. The three point cover $9b$ has ramification points, $\sqrt{-3}$, $-\sqrt{-3}$, ∞ rather than the usual 0, 1, ∞.

N	λ_0	λ_1	λ_∞	g	$\mathrm{Res}(t)$	Bad p		
$9a$	333	22221	9	1	$\mathbf{Q}(\cos(2\pi/9))$	2	3	
$9b$	(33111	33111)	9	0	$\mathbf{Q}[x]/r_{9b}(t,x)$	2	3	
$9c$	333	22221	711	0	$\mathbf{Q}(\cos(2\pi/9))$	2	3	7

$$f_{9a}(t,x) = (x^3 - 9x^2 - 69x - 123)^3 - \\ 2^{14}t(9x^4 - 42x^3 - 675x^2 - 1485x - 441) - 2^{28}t^2,$$

$$f_{9b}(t,x) = x^9 + 108x^7 + 216x^6 + 4374x^5 + 13608x^4 + 99468x^3 + \\ 215784x^2 + 998001x + 663552t + 810648,$$

$$f_{9c}(t,x) = 4(x^3 + 4x^2 + 10x + 6)^3 - 27t(4x^2 + 13x + 32).$$

Cover 9a was originally found by Elkies and Cover 9b by Matzat. For more on both, see [21, §6]. In particular for Cover 9a, some of the fields on Table 9.3 come from even more specialization points than listed on Table 9.3, and this phenomenon is explained in large part by isogenies, like in Section 8. The cubic subfield associated to $9b$ is the only one which varies with t, it being given by

$$r_{9b}(t,x) = x^3 - (9t^2 + 27)x - (9t^3 + 9t^2 + 27t + 27).$$

This cover is the only one of the three which has specializations with Galois group $SL_2(8)$, such as those given in [21, §6]. However Cover $9b$ is not a very fecund source for $SL_2(8)$ fields, as it seems all the $SL_2(8)$ fields it gives have slope data $2[2,2,2]$ at 2. At any rate, we did not find any $SL_2(8)$ fields with GRD less than Ω from $9b$. Note finally that non totally real fields with Galois group $\Sigma L_2(8)$ should correspond to Hilbert modular forms on the totally real cubic subfield.

10. DEGREE 10

In this section we focus on the non-solvable group $T35 = P\Gamma L_2(9) = PSL_2(9).2^2$ and its index two subgroups $T30 = PGL_2(9)$ and $T31 = M_{10}$. Another interesting class of non-solvable decic groups, A_5^2 and its index 2, 4, and 8 overgroups, will be treated in Section 12.

Table 10.1 shows seven triples $(\lambda_0, \lambda_1, \lambda_\infty)$ of partitions of ten. All of them give rise to decic three point covers with Galois group $P\Gamma L_2(9)$, equations being

TABLE 9.3. The fifteen known fields in $\mathcal{K}(\Sigma L_2(8), \Omega)$. All of them were found by specializing three point covers.

GRD	Polynomial	Sources	Slope data
34.36	$x^9 - 3x^8 + 4x^7 + 16x^2 + 8x + 8$	c $7^3/2^1 3^3$	$2\left[\frac{20}{7}, \frac{20}{7}, \frac{20}{7}\right]$ 7_9
35.52	$x^9 - x^8 + 2x^7 + 28x^5 - 28x^4$ $+28x^3 + 24x^2 + 200x - 204$	c $5^1 7^5/2^{13}$	$2\left[\frac{8}{7}, \frac{8}{7}, \frac{8}{7}\right]$ 5_3 7_9
35.72	$x^9 - 6x^6 - 12x^3 - 36x^2$ $-18x - 4$	a -2^3 a $-3^2/2^4$	$2\left[\frac{12}{7}, \frac{12}{7}, \frac{12}{7}\right]$ $3\left[\frac{3}{2}, 2, \frac{5}{2}\right]$
36.12	$x^9 - 3x^8 + 12x^7 - 20x^6 + 36x^5$ $-36x^4 + 40x^3 - 24x^2 + 12x - 4$	a $11^3/2^3$	$2\left[\frac{18}{7}, \frac{18}{7}, \frac{18}{7}\right]$ $3[2, 2]$
37.18	$x^9 - 6x^6 - 18x^5 - 54x^4 - 90x^3$ $-90x^2 - 54x - 16$	a 2^2 a $-2^4 3$	$2\left[\frac{8}{7}, \frac{8}{7}, \frac{8}{7}\right]$ $3[2, 2, 3]$
37.57	$x^9 - 24x^6 + 48x^3 + 216x^2$ $-108x - 296$	a $-11^3/2^8$ a $3/2^2$	$2\left[\frac{10}{7}, \frac{10}{7}, \frac{10}{7}\right]$ $3[2, 3]$
40.18	$x^9 - 3x^8 + 6x^7 - 10x^6 + 12x^5$ $-12x^4 + 8x^3 - 12x^2 - 4$	a $-5^3/2^6$	$2\left[\frac{8}{7}, \frac{8}{7}, \frac{8}{7}\right]$ $3[2, 2]$ 7_2
40.41	$x^9 - 3x^8 + 12x^6 - 6x^5 - 18x^4$ $+48x^3 - 84x^2 + 63x - 17$	a $1/2$	$2\left[\frac{20}{7}, \frac{20}{7}, \frac{20}{7}\right]$ $3\left[\frac{3}{2}, 2\right]$
41.78	$x^9 - 3x^8 + 6x^7 + 8x^6 - 24x^5$ $+42x^4 + 6x^3 + 30$	a $-2^4/5$ a $2^2/5^3$	$2\left[\frac{8}{7}, \frac{8}{7}, \frac{8}{7}\right]$ $3\left[\frac{3}{2}, 2\right]$ 5_3
41.79	$x^9 - x^8 - 4x^7 + 28x^3 + 26x^2$ $+9x + 1$	c $-5^3/2^6$ c $2^2 7/3^3$	$2\left[\frac{8}{7}, \frac{8}{7}, \frac{8}{7}\right]$ $7\left[\frac{5}{3}\right]$
41.90	$x^9 - 3x^8 + 4x^7 + 6x^2 + 3x + 3$	c $2^5/3^4$ c $2^7 7^2/3^8$	2_9 3_7 $7\left[\frac{7}{6}\right]$
41.98	$x^9 - 3x^8 + 4x^7 - 8x^2 - 4x - 4$	c $5^3/3^3$ c $-7^2 17^3/3^3 5^7$	$2[2, 2, 3]$ $7\left[\frac{7}{6}\right]$
42.96	$x^9 - 12x^5 - 24x^4 + 16x^3 + 48x^2$ $+12x + 16$	b 0	$2\left[\frac{20}{7}, \frac{20}{7}, \frac{20}{7}\right]$ $3[2, 2]$
43.69	$x^9 - 3x^8 - 24x^6 + 18x^5 + 18x^4$ $-24x^3 + 9x - 3$	a 3^2 a $5^3/3^3$	$2[2, 2, 3]$ $3\left[\frac{3}{2}, 2, \frac{13}{6}\right]$
44.68	$x^9 - 24x^6 + 18x^5 + 144x^3$ $-216x^2 + 81x + 24$	a $-2^3 3$ a $1/2^2$	$2\left[\frac{12}{7}, \frac{12}{7}, \frac{12}{7}\right]$ $3[2, 3]$

$$f_{10a}(t, x) = 4(x^3 + 6x^2 + 15x + 12)^3 x + 27t(3x^2 + 14x + 27),$$
$$f_{10b}(t, x) = x^8(x - 3)^2 - 27t(3x^2 - 2x + 3),$$

$$f_{10c}(t, x) = (5x^2 - 81)^4(5x^2 + 50x + 189) - 2^{14}3^{12}t,$$

$$f_{10d}(t, x) = (15x^2 + 10x + 3) + 12tx^5(5x + 2) + 64t^2x^{10},$$

TABLE 10.1. Information corresponding to seven decic three point covers with generic Galois group $P\Gamma L_2(9)$.

N	λ_0	λ_1	λ_∞	g	S_6	$PGL_2(9)$	M_{10}	Bad p		
$10a$	3331	22222	811	0	$-(t-1)$	2	$-2(t-1)$	2	3	
$10b$	82	2221111	811	0	t	$2t(t-1)$	$2(t-1)$	2	3	
$10c$	4411	2221111	(10)	0	t	$5t(t-1)$	$5(t-1)$	2	3	5
$10d$	4411	222211	55	0	-15	5	-3	2	3	5
$10e$	3331	22222	(10)	1	$-(t-1)$	5	$-5(t-1)$	2	3	5
$10f$	442	22222	82	1	$-5(t-1)$	$10t$	$-2t(t-1)$	2		5
$10g$	4411	22222	(10)	1	$-3(t-1)$	5	$-15(t-1)$	2	3	5

$$f_{10e}(t,x) = (1-t)(x^3-60x-200)^3(x-20)+ \\ t(x^5-10x^4-140x^3+100x^2+2600x-14504)^2+t(t-1)2^6 3^{14},$$

$$f_{10f}(t,x) = 16(1-t)x^2(x^2+5x+5)^4+ \\ t(4x^5+40x^4+140x^3+200x^2+105x+34)^2+(t-1)t(5x+2)^2,$$

$$f_{10g}(t,x) = (1-t)(x^2-15)^4(x^2+20x+180)+ \\ t(x^5+10x^4+10x^3-700x^2+2225x-3046)^2+t(t-1)2^8 3^8.$$

For $N = 10a$, $10b$, $10c$, our polynomial $f_N(t,x)$ presents Cover N in the usual simple form $\mathbf{P}^1_x \to \mathbf{P}^1_t$. Cover $10d$ is also of genus zero, but it is more complicated as it has no rational points, in fact no real points and no points over $\mathbf{Q}_5$. Covers $10e$, $10f$, and $10g$ all have genus one. The Galois group of the number field $K_{N,t}$ is in H if the quantity printed in the (N,H) slot in Table 10.1 is a square in $\mathbf{Q}^\times$.

Table 10.2 gives the sub-Ω fields we found. Although all covers except for $10d$ are capable of producing M_{10} fields, we did not find any sub-Ω M_{10} fields. The covers also reproduced some of the A_6 and S_6 fields on Tables 5.2 and 5.3.

A form numerically matching the first field on Table 10.2 is

$$\begin{aligned} f_{38.52} =& (2-\sqrt{2})\hat\theta_1\theta_{16}\theta_{32}\check\theta_{256}+(\sqrt{2}-1)\hat\theta_1\theta_8\theta_{16}\check\theta_{256} \\ &+2i\sqrt{2+\sqrt{2}}\hat\theta_1\phi_2\theta_{32}\check\theta_{256}-i\sqrt{4+2\sqrt{2}}\hat\theta_1\phi_2\theta_8\check\theta_{256} \\ &-2i\phi_1\phi_4\theta_{16}\theta_{32}+i\sqrt{2}\phi_1\phi_4\theta_8\theta_{16} \\ &-2\sqrt{4-2\sqrt{2}}\phi_1\phi_2\phi_4\theta_{32}-\sqrt{8-4\sqrt{2}}\phi_1\phi_2\phi_4\theta_8 \in S_2^{\mathrm{prim}}(1024,\chi_8). \end{aligned} \tag{10.1}$$

To get matches for the $P\Gamma L_2(9)$ fields, one would have to work with GL_2 over the quadratic subfield corresponding to $PGL_2(9)$, thus holomorphic Hilbert modular forms in the case this subfield is real.

TABLE 10.2. The two known fields in $\mathcal{K}(PGL_2(9),\Omega)$ and the fourteen known fields in $\mathcal{K}(P\Gamma L_2(9),\Omega)$

GRD	Polynomial	Sources	Slope data
36.79	$x^{10}-3x^8+12x^6-24x^5$ $+12x^4+45x^2-24x+9$	b -1	$2[2,3,4]$ $3\left[\frac{3}{2},\frac{3}{2}\right]$
43.91	$x^{10}-2x^9+9x^8-7x^2$ $+14x-7$	b $3^6/7$	$2[2,3,4]$ 7_8
38.61	$x^{10}-4x^9+6x^8-9x^6+3x^4$ $+18x^3-12x+6$	d $5^3/2^2$	$2[2,2]$ $3\left[\frac{9}{8},\frac{9}{8}\right]$ 5_8
39.28	$x^{10}-6x^8+9x^6-12x^5+24x^4$ $+21x^2+4x+18$	a 2^2	$2[2,3,3]$ $3\left[\frac{15}{8},\frac{15}{8}\right]$
40.12	$x^{10}-2x^9+3x^8-12x^6+24x^5$ $-12x^4+21x^2-74x+55$	a $5^3/2^2$ a $15613^3/3^8 11^8$	$2[2,2,3,4]$ $3\left[\frac{3}{2},\frac{3}{2}\right]$
40.45	$x^{10}-5x^9+15x^8-30x^7+45x^6$ $-51x^5+30x^4-15x^2-15x+42$	e $2^5/3^2$ d $3^3 5/2^8$	$3\left[\frac{9}{8},\frac{9}{8}\right]$ $5\left[\frac{7}{4}\right]$
41.19	$x^{10}-3x^8+18x^6-30x^4+39x^2$ $-16x+3$	a $5^3/3^3$	$2\left[2,3,\frac{7}{2},\frac{9}{2}\right]$ $3\left[\frac{9}{8},\frac{9}{8}\right]$
41.65	$x^{10}-2x^9+15x^6+12x^5+18x^4$ $+12x^3+12x^2+8x+2$	a 2^6	$2\left[\frac{4}{3},\frac{4}{3},3\right]$ $3\left[\frac{5}{4},\frac{5}{4}\right]$ 7_2
42.83	$x^{10}-2x^9+3x^8+9x^6-18x^5$ $+9x^4+27x^2-18x+9$	a $13^3/2^2 3^5$	$2[2,2,3,3]$ $3\left[\frac{15}{8},\frac{15}{8}\right]$
43.01	$x^{10}-4x^9+6x^8+24x^2$ $+32x+16$	a $11^3/2^3$	$2[2,3,4,5]$ 3_{10}
43.52	$x^{10}+9x^8+6x^6-30x^4-48x^3$ $-36x^2-16x-12$	a $-5^3/3^7$	$2[2,3,4]$ $3\left[\frac{13}{8},\frac{13}{8}\right]$
43.87	$x^{10}-2x^9+3x^8-6x^6+12x^5$ $-30x^4-15x^2-2x-41$	a $-5^3/2^6$	$2[2,2,3,3]$ 3_{10} 7_2
43.93	$x^{10}-5x^8-10x^7+10x^6+8x^5$ $+10x^4+50x^3+85x^2-60x+1$	e $2/3^3$	$2[2]_5$ 3_8 $5\left[\frac{5}{4}\right]$
44.09	$x^{10}-4x^9+6x^8-12x^2$ $-16x-8$	a $13^3/2^4 5^4$	$2\left[2,2,3,\frac{7}{2}\right]$ 3_{10} 5_2
44.20	$x^{10}-15x^7-27x^5+75x^4$ $+135x^2+10x+81$	e 2^2	$3\left[\frac{15}{8},\frac{15}{8}\right]$ $5\left[\frac{5}{4}\right]$
44.74	$x^{10}-4x^9+12x^8-24x^7$ $+39x^6-48x^5+48x^4-36x^3$ $+24x^2-12x+6$	a $2^5/3^3$	$2\left[\frac{8}{3},\frac{8}{3},3\right]$ $3\left[\frac{9}{8},\frac{9}{8}\right]$ 5_2

11. DEGREE 11

In degree 11, we know of only one non-solvable sub-Ω field, the splitting field K of either

$$f_{11a}(x) = x^{11}-2x^{10}+3x^9+2x^8-5x^7+16x^6-10x^5+10x^4+ 2x^3-3x^2+4x-1,$$

$$f_{11b}(x) = x^{11} - 2x^{10} + x^9 - 5x^8 + 13x^7 - 9x^6 + x^5 - 8x^4 + 9x^3 - 3x^2 - 2x + 1.$$

One has $\mathrm{Gal}(K/\mathbf{Q}) \cong PSL_2(11)$ and $d(K) = 1831^{1/2} \approx 42.79$. Here $\mathbf{Q}[x]/f_{11a}(x)$ and $\mathbf{Q}[x]/f_{11b}(x)$ are twin fields, corresponding to two non-conjugate embeddings of the order 60 group A_5 into the order 660 group $PSL_2(11)$.

The polynomial $f_{11a}(x)$ appears at [14]. Here we sketch the resolvent construction we used to compute $f_{11b}(x)$ from $f_{11a}(x)$. Let α_j be the complex roots of $f_{11a}(x)$ with approximate values as follows:

			4	6		9	11
j	1	2			7		
			3	5		8	10
α_j	-1.6	-0.7	$0.1 \pm 1.0i$	$0.2 \pm 1.2i$	0.3	$0.4 \pm 0.7i$	$1.3 \pm 1.3i$

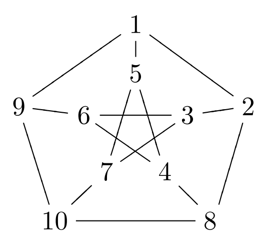
An element of $\mathrm{Gal}(K/\mathbf{Q})$ is complex conjugation

$$\sigma = (3,4)(5,6)(8,9)(10,11). \tag{11.1}$$

We need to complement this element with others to obtain all of $\mathrm{Gal}(K/\mathbf{Q})$.

The degree 165 polynomial with roots $\alpha_i + \alpha_j + \alpha_k$ factors over $\mathbf{Z}$ into irreducibles as $f_{55}(x)f_{110}(x)$. Say that $i, j \in \{1, \ldots, 10\}$ are adjacent iff $\alpha_i + \alpha_j + \alpha_{11}$ is a root of the degree 55 polynomal $f_{55}(x)$. This definition yields the valence 3 graph P in Figure 11.1 (called the Petersen graph, see e.g. [25]). The automorphism group $A(P)$ of this graph is isomorphic to S_5. Its alternating subgroup $A(P)^+$ is the subgroup of $\mathrm{Gal}(K/\mathbf{Q})$ fixing α_{11}. Since $A(P)^+$ is a maximal subgroup of $PSL_2(11)$ and $\sigma \notin A(P)^+$, σ and $A(P)^+$ together generate all of $\mathrm{Gal}(K/\mathbf{Q})$.

FIGURE 11.1. Adjacency relations among $\alpha_1, \ldots, \alpha_{10}$.



An element of $A(P)^+$ evident from Figure 11.1 is "clockwise rotation by 1/5 turn," or

$$r = (1,2,8,10,9)(5,3,4,7,6). \tag{11.2}$$

Let H be the subgroup $\langle \sigma, r \rangle$ of $\mathrm{Gal}(K/\mathbf{Q})$. One can check that it is a transitive subgroup with sixty elements. Let C be the set of right cosets of H in $\mathrm{Gal}(K/\mathbf{Q})$. For $c \in C$, define

$$\beta_c = \sum_{g \in c} \alpha_{g1}\alpha_{g2}\alpha_{g3}.$$

Define $F_{11b}(x) \in \mathbf{Z}[x]$ to be the monic degree 11 polynomial with roots β_c. Our polynomial $f_{11b}(x)$ is obtained by applying *Pari*'s `polredabs` to $F_{11b}(x)$.

12. Composita

Let G be a group with surjections $i_1 : G \to Q_1$ and $i_2 : G \to Q_2$ such that $(i_1, i_2) : G \to Q_1 \times Q_2$ is an injection. Then any field $K \in \mathcal{K}(G, C)$ is a compositum $K_1 K_2$ with $K_i \in \mathcal{K}(Q_i, C)$.

We have looked at all composita $K = K_1 K_2$ with each K_i different fields in one of the identified sets $\mathcal{K}(Q, \Omega)$ with $Q = A_5$, S_5, A_6, or S_6. In all cases, the root discriminant of K is above Ω, proving the following result.

Corollary 12.1. *Let G be one of the twelve groups $A_m \times A_n$, $A_m \times S_n$, $S_m \times_2 S_n$ or $S_m \times S_n$ with $(m, n) = (5, 5)$, $(5, 6)$, or $(6, 6)$. Then $\mathcal{K}(G, \Omega)$ is empty.*

Several times, basic invariants of K_1 and K_2 allowed the possibility that $K_1 K_2$ might be sub-Ω, but in every such case computation of more refined invariants revealed that $K_1 K_2$ has root discriminant above Ω. One such instance is the following. Consider

$$\begin{aligned} f_1(x) &= x^5 - 2x^4 + 13x^3 - 9x^2 + 36x - 12 \\ f_2(x) &= x^5 + 19x^2 - 57, \end{aligned}$$

both of which have Galois group A_5. Both splitting fields K_i have root discriminant $3^{7/6} 19^{4/5} \approx 37.9878$. At issue is the root discriminant $3^{\beta_3} 19^{\beta_{19}}$ of $K_1 K_2$. At 19, both K_1 and K_2 are tame with $t = 5$, so $\beta_{19} = 4/5$. At 3, both root fields $\mathbf{Q}[x]/f_i(x)$ factor 3-adically as a wildly ramified cubic with unique wild slope $3/2$ times a tamely ramified quadratic. This leaves two possibilities, as the cubic and the quadratic could both have discriminant 3 or both -3 in $\mathbf{Q}_3^\times / \mathbf{Q}_3^{\times 2}$. For f_1 this discriminant is -3 while for f_2 this discriminant is 3. This causes $K_1 K_2$ to have slope of $3/2$ with multiplicity two, rather than one. So β_3 is $(8/9)(3/2) + (1/9)(1/2) = 25/18$ rather than $(2/3)(3/2) + (1/3)(1/2) = 7/6$. So the root discriminant of $K_1 K_2$ is $3^{25/18} 19^{4/5} \approx 48.4921$.

Consider next groups $G = A_m^2.H \subseteq S_{2m}$, with $m = 5, 6$ and H a subgroup of D_4. We have just considered the cases with H not switching the two factors of A_m. The remaining cases correspond to $C_2 \subseteq H \subseteq D_4$, i.e. $H = C_2$, V, C_4, and D_4. In the case $m = 5$, these groups are T40, T41, T42, and T43 respectively, while for $m = 6$ they are T296, T297, T298, and T299 respectively. Given Corollary 12.1, one might expect that there are very few sub-Ω fields for these groups. If fact, we have found only one, the splitting field for

$$x^{10} + 2x^8 - 8x^7 - 8x^6 - 16x^5 - 16x^4 - 8x^3 - 14x^2 - 4, \tag{12.1}$$

with Galois group $A_5^2.2$, slope data $[2, 2, 3, 7/2, 7/2]$ at 2 and $[3/2, 3/2]$ at 3, and root discriminant $2^{51/16} 3^{25/18} \approx 41.90$.

The way we first encountered the splitting field of (12.1) was as follows. We considered the family $x^6 + 2ax^3 + 3bx^2 + c$ which has polynomial discriminant $2^6 3^6 c(a^4 b^3 - 16b^6 + a^6 c - 20a^2 b^3 c - 3a^4 c^2 - 8b^3 c^2 + 3a^2 c^3 - c^4)$. Plugging in a, b, and c all from the same quadratic field will generically give all of $A_6^2.D_4$ as a splitting field over $\mathbf{Q}$. However we found $a = -b = c = -1 + \sqrt{2}$ multiplied with its conjugate gives a degree twelve polynomial with the above Galois root discriminant. Its Galois group is not the 1,036,800-element group $A_6^2.D_4$, but rather the 7,200-element group $T269 = PGL_2(5)^2.2$. Twinning down over $\mathbf{Q}(\sqrt{2})$, we get the decic polynomial (12.1).

13. Larger degrees

In the previous sections, we presented non-solvable sub-Ω fields with Galois group involving the first through fifth and then eighth simple group, in order of size, as listed in Section 1. The sixth and seventh groups are $PSL_2(13)$ and $PSL_2(17)$. The ninth through twenty-first groups are $PSL_2(19)$, $SL_2(16)$, $SL_3(3)$, $SU_3(3)$, $PSL_2(23)$, $PSL_2(25)$, M_{11}, $PSL_2(27)$, $PSL_2(29)$, $PSL_2(31)$, A_8, $PSL_3(4)$, and $SU_4(2)$. We have specialized at least one three point cover corresponding to these fifteen groups, and more three point covers for larger simple groups as well, but have not found any corresponding sub-Ω fields.

Here we will report on just one of the computations that did not result in a sub-Ω field. We choose this one because we think it gives the best candidate for a minimal root discriminant d_G. Otherwise, the computation is quite representative of the others we have done.

Our group is $\Sigma L_2(16) \cong SL_2(16).4$ with $SL_2(16)$ having order $4080 = 2^4 \cdot 3 \cdot 5 \cdot 17$. The first necessity is to find corresponding degree $|\mathbf{P}^1(16)| = 17$ three point covers, as this is the smallest degree group omitted from Table 10 in the appendix of the standard reference [17]. The most promising class triples are $(3A, 2A, 15ABCD)$ and $(4A, 2B, 15ABCD)$, corresponding to partition triples $(3^5 1^2, 2^8 1, 15 \cdot 1^2)$ and $(4^4 1, 2^6 1^5, 15 \cdot 1^2)$ respectively. We find the corresponding covers to be

$$\begin{aligned} f_1(t,x) &= 2^2(x^5+3x^4+12x^3+18x^2+27x+9)^3(x^2+3x+6) - \\ &\quad t3^6(4x^2+3x+24), \end{aligned}$$

$$\begin{aligned} f_2(t,x) &= 3^3(x^4+2x^3+4x^2+28x-4)^4(x-2) + \\ &\quad t2^{12}5^5(2x^2-3x+18), \end{aligned}$$

with discriminants $D_1(t) = 2^{60}3^{124}5^{18}t^{10}(t-1)^8$ and $D_2(t) = 2^{232}3^{60}5^{106}t^{12}(t-1)^6$. The generic Galois group is indeed $SL_2(16).4$ in both cases. However the monodromy group is $SL_2(16)$ and $SL_2(16).2$ in the two cases. The quartic extensions of $\mathbf{Q}(t)$ corresponding to the .4 are

$$\begin{aligned} g_1(t,x) &= x^4 - x^3 - 4x^2 + 4x + 1, \\ g_2(t,x) &= x^4 - 15t^2x^2 + 15tx^2 + 45t^4 - 90t^3 + 45t^2. \end{aligned}$$

The first of these comes from the constant field extension $\mathbf{Q}(\zeta_{1/15})^+$ of $\mathbf{Q}$. The second gives quartic C_4 fields varying with t, but all containing $\mathbf{Q}(\sqrt{5})$ and all with discriminant exactly divisible by 5^3.

A specialization point $\tau \in \mathbf{Q} - \{0,1\}$ gives a degree 17 algebra $\mathbf{Q}[x]/f_i(\tau, x)$. To keep the discriminant of this algebra of the form $2^\alpha 3^\beta 5^\gamma$, a necessary and sufficient condition is that τ can be written in the form $-ax^p/cz^r$ with

$$ax^p + by^q + cz^r = 0,$$

a, b, c integers with all prime factors in $\{2,3,5\}$, and x, y, z integers. Here (p,q,r) is $(3,2,15)$ for the first cover and $(4,2,15)$ for the second.

We found over 400 such specialization points for f_1 and over 200 for f_2. Exactly three points for f_1 and two points for f_2 gave a Galois root discriminant less than Ω. These are indicated in Table 13.1. As one can tell from the three specialization points for f_1, ramification can be made tame at any of 2, 3, and 5, Similarly one can see from either of the two specialization points that 3 can even be made

TABLE 13.1. The three specialization points τ for f_1 and the two for f_2 giving sub-Ω Galois root discriminants. All five polynomials $f_i(\tau, x)$ factor as an irreducible degree d polynomial times an irreducible polynomial of degree $17-d$, showing immediately that the Galois group G is not all of $SL_2(16).4$. In each case, the Galois group is given, and also a polynomial with degree $< d$ and the same splitting field.

τ	GRD	d	G	Another defining polynomial
$-2^7 5/3^6$	$2^{4/5}3^{1/2}5^{31/20} \approx 36.54$	16	$2^4.F_5$	$x^{10}-20x^6+80x^2-16$
$5^2 17^3/2^{10}$	$2^{7/6}3^{4/5}5^{23/20} \approx 38.41$	12	$S_5 \times_2 4$	$(x^5+15x-6)\cdot$ $(x^4-x^3-4x^2+4x+1)$
$2^7/3$	$2^{4/5}3^{11/6}5^{3/4} \approx 43.63$	12	$S_5 \times_2 4$	$(x^5-2x^4+4x^3+2x^2-4x-10)\cdot$ $(x^4-x^3-4x^2+4x+1)$
$-3^3 5^2$	$2^2 5^{71/60} \approx 26.86$	15	$F_5 \times S_3$	$(x^5+5x^3+5x-2)\cdot$ (x^3-x^2+2x+2)
$3^3 5/2^8$	$2^{7/6}5^{31/20} \approx 27.20$	12	$S_5 \times_2 4$	$(x^5-10x^2-10x-16)\cdot$ $(x^4-x^3+x^2-x+1)$

unramified for f_2. Accordingly, we also tried more specialization points for each cover, allowing other primes to ramify, but did not find any more which gave a Galois root discriminant less than Ω.

As explained in some detail on the table, our five specialization points all gave a field with Galois group not containing $SL_2(16)$. This phenomenon was seen very often while specializing our other covers. It, and similar behaviors such as that described in the last paragraph of Section 12, give support to the expectation that there are very few sub-Ω fields for the groups in question.

The smallest GRD we found for a field with Galois group containing $SL_2(16)$ is

$$2^{101/60}3^{3/4}5^{23/20} \approx 46.60,$$

coming from the splitting field of $f_1(-8, x)$ with slope data

$$\left(2\left[\frac{26}{15}, \frac{26}{15}, \frac{26}{15}, \frac{26}{15}\right],\ 3_4,\ 5\left[\frac{5}{4}\right]\right). \tag{13.1}$$

and Galois group all of $\Sigma L_2(16)$. A polynomial with smaller coefficients having the same splitting field is

$$\begin{aligned} f(x) \quad = \quad & x^{17}-x^{16}+4x^{15}+20x^{12}-20x^{11}+20x^{10}+10x^9-50x^8+ \\ & 44x^7-24x^6+16x^5+20x^4-60x^3-12x^2-3x-13. \end{aligned}$$

The derivation of (13.1) requires more than our usual mechanical appeal to our database at 2 and 5, so we sketch our procedure here for 2, the harder of the two cases. We begin by factoring $f(x)$ over $\mathbf{Q}_2$, getting a totally ramified degree 16 factor with discriminant 2^{26} and a degree 1 factor. Let $\alpha_1 \in \mathbf{Q}_2$ be the root of the degree one factor and let $\alpha_2, \ldots, \alpha_{17}$ be the roots of the degree 16 factor in an algebraic closure of $\mathbf{Q}_2$. Consider the polynomial $d(x) \in \mathbf{Z}[x]$ with roots $\alpha_i - \alpha_j$. We calculate this algebraically and then take its 2-adic Newton polygon, finding it to have slopes $3/16$ with multiplicity 32 and $7/30$ with multiplicity 240. The $3/16$ can only be the 2-adic valuation of the roots $\alpha_i - \alpha_j$ with $1 \in \{i, j\}$ and the $7/30$ must be the 2-adic valuation of the remaining roots. The odd part of 30 is 15, so 15 must divide, hence be, the size of the tame inertia group. This forces there to be four wild slopes, all equal, hence all $26/15$.

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